2019

AP® CollegeBoard

AP® Calculus BC

Sample Student Responses and Scoring Commentary

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Question 1

(a) $\int_0^5 E(t)\,dt = 153.457690$

To the nearest whole number, 153 fish enter the lake from midnight to 5 A.M.

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(b) $\frac{1}{5-0}\int_0^5 L(t)\,dt = 6.059038$

The average number of fish that leave the lake per hour from midnight to 5 A.M. is 6.059 fish per hour.

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(c) The rate of change in the number of fish in the lake at time t is given by $E(t) - L(t)$.

$E(t) - L(t) = 0 \Rightarrow t = 6.20356$

$E(t) - L(t) > 0$ for $0 \le t < 6.20356,$ and $E(t) - L(t) < 0$ for $6.20356 < t \le 8.$ Therefore the greatest number of fish in the lake is at time $t = 6.204$ (or 6.203).

— OR —

Let $A(t)$ be the change in the number of fish in the lake from midnight to t hours after midnight.

$A(t) = \int_0^t (E(s) - L(s))\,ds$

$A'(t) = E(t) - L(t) = 0 \Rightarrow t = C = 6.20356$

t	$A(t)$
0	0
C	135.01492
8	80.91998

Therefore the greatest number of fish in the lake is at time $t = 6.204$ (or 6.203).

$3 : \begin{cases} 1 : \text{sets } E(t) - L(t) = 0 \\ 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$

(d) $E'(5) - L'(5) = -10.7228 < 0$

Because $E'(5) - L'(5) < 0,$ the rate of change in the number of fish is decreasing at time $t = 5.$

$2 : \begin{cases} 1 : \text{considers } E'(5) \text{ and } L'(5) \\ 1 : \text{answer with explanation} \end{cases}$

1. Fish enter a lake at a rate modeled by the function E given by $E(t) = 20 + 15\sin\left(\frac{\pi t}{6}\right)$. Fish leave the lake at a rate modeled by the function L given by $L(t) = 4 + 2^{0.1t^2}$. Both $E(t)$ and $L(t)$ are measured in fish per hour, and t is measured in hours since midnight ($t = 0$).

(a) How many fish enter the lake over the 5-hour period from midnight ($t = 0$) to 5 A.M. ($t = 5$) ? Give your answer to the nearest whole number.

$\int_0^5 E(t)\,dt \approx 153$ fish

(b) What is the average number of fish that leave the lake per hour over the 5-hour period from midnight ($t = 0$) to 5 A.M. ($t = 5$) ?

$\frac{1}{5}\int_0^5 L(t)\,dt \approx \frac{30.295}{5} \approx 6.059$ fish per hour leave the lake

Continue question 1 on page 5.

(c) At what time t, for $0 \le t \le 8$, is the greatest number of fish in the lake? Justify your answer.

$E(t)-L(t)=0$

$t \approx 6.204$

$E(t)-L(t)$ is the rate at which the number of fish is changing

At time $t=6.204$, the greatest number of fish in the 8 hour period are in the lake. This is because $E(t)-L(t)$ is positive from $t=0$ to $t=6.204$ indicating that the number of fish in the lake is increasing over $(0, 6.204)$, but $E(t)-L(t)$ is negative from $t=6.204$ to $t=8$, which means the number of fish are decreasing in this time period, so the number of fish in the lake is greatest at $t=6.204$ hours

(d) Is the rate of change in the number of fish in the lake increasing or decreasing at 5 A.M. ($t = 5$)? Explain your reasoning.

$\frac{d}{dt}\left(16+15\sin\left(\frac{\pi t}{6}\right)-2^{0.1t^2}\right)\Big|_{t=5} \approx -10.723$ fish/hour^2

Since the derivative of $E(t)-L(t)$ at $t=5$ is negative, the rate of change in the number of fish in the lake is decreasing

1. Fish enter a lake at a rate modeled by the function E given by $E(t) = 20 + 15\sin\left(\frac{\pi t}{6}\right)$. Fish leave the lake at a rate modeled by the function L given by $L(t) = 4 + 2^{0.1t^2}$. Both $E(t)$ and $L(t)$ are measured in fish per hour, and t is measured in hours since midnight ($t = 0$).

(a) How many fish enter the lake over the 5-hour period from midnight ($t = 0$) to 5 A.M. ($t = 5$) ? Give your answer to the nearest whole number.

$$\int_0^5 E(t)\,dt = 153.458$$

153 fish enter over 5-hour period

(b) What is the average number of fish that leave the lake per hour over the 5-hour period from midnight ($t = 0$) to 5 A.M. ($t = 5$) ?

$$\frac{1}{5}\int_0^5 L(t)\,dt = 6.059$$

Continue question 1 on page 5.

(c) At what time t, for $0 \le t \le 8$, is the greatest number of fish in the lake? Justify your answer.

max

$E(t) - L(t) = 0$ and changes signs (+) to (−)

$t = 6.204$

There is the greatest number of fish in the lake at time $t = 6.204$ hours

(d) Is the rate of change in the number of fish in the lake increasing or decreasing at 5 A.M. ($t = 5$) ? Explain your reasoning.

$E(5) - L(5) = 17.843$

The rate of change in the number of fish in the lake is increasing since $E(5) - L(5) > 0$ and represents the rate of change.

1. Fish enter a lake at a rate modeled by the function E given by $E(t) = 20 + 15\sin\left(\frac{\pi t}{6}\right)$. Fish leave the lake at a rate modeled by the function L given by $L(t) = 4 + 2^{0.1t^2}$. Both $E(t)$ and $L(t)$ are measured in fish per hour, and t is measured in hours since midnight ($t = 0$).

(a) How many fish enter the lake over the 5-hour period from midnight ($t = 0$) to 5 A.M. ($t = 5$)? Give your answer to the nearest whole number.

$\int_0^5 \left(20+15\sin\left(\frac{\pi t}{6}\right)\right)dt$ ~~$\int_0^5 4+2^{0.1t^2}$~~

153.458 or $\boxed{153}$

(b) What is the average number of fish that leave the lake per hour over the 5-hour period from midnight ($t = 0$) to 5 A.M. ($t = 5$)?

$\int_0^5 \left(4+2^{0.1t^2}\right)dt \rightarrow \boxed{30.295}$

Continue question 1 on page 5.

1C
2 of 2

(c) At what time t, for $0 \le t \le 8$, is the greatest number of fish in the lake? Justify your answer.

~~8~~ 5 because $\int_0^8 \left(20 + 15\sin\left(\frac{\pi t}{6}\right)\right)dt \int_0^8 \left(4 + 2^{0.1t^2}\right) dt$ 80.92

and $\int_0^5 \left(20 + 15\sin\left(\frac{\pi t}{6}\right)\right)dt \int_0^8 \left(4 + 2^{0.1t^2}\right) dt =$ 123.163

123.163 > 80.92

(d) Is the rate of change in the number of fish in the lake increasing or decreasing at 5 A.M. ($t = 5$)? Explain your reasoning.

Increasing because at 4 A.M. 100

$\left(\int_0^4 \left(20 + 15\sin\left(\frac{\pi t}{6}\right)\right)dt \int_0^4 \left(4 + 2^{0.1t^2}\right) dt = 100.837\right)$

fish are in the lake but at

5 A.M. → 123.163 fish are in the

lake $\left(\int_0^5 \left(20 + 15\sin\left(\frac{\pi t}{6}\right)\right)dt - \int_0^5 \left(4 + 2^{0.1t^2}\right) dt = 123.163\right)$

123.163 > 100.837

Question 1

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

In this problem, fish enter and leave a lake at rates modeled by functions E and L given by $E(t) = 20 + 15\sin\left(\frac{\pi t}{6}\right)$ and $L(t) = 4 + 2^{0.1t^2}$, respectively. Both $E(t)$ and $L(t)$ are measured in fish per hour, and t is measured in hours since midnight $(t = 0)$.

In part (a) students were asked to find the number of fish entering the lake between midnight $(t = 0)$ and 5 A.M. $(t = 5)$ and to provide the answer rounded to the nearest whole number. A response should demonstrate an understanding that a definite integral of the rate at which fish enter the lake over the time interval $0 \le t \le 5$ gives the number of fish that enter the lake during that time period. The numerical value of the integral $\int_0^5 E(t)\,dt$ should be obtained using a graphing calculator.

In part (b) students were asked for the average number of fish that leave the lake per hour over the 5-hour period $0 \le t \le 5.$ A response should demonstrate that "number of fish per hour" is a rate, so the question is asking for the average value of $L(t)$ across the interval $0 \le t \le 5,$ found by dividing the definite integral of L across the interval by the width of the interval. The numerical value of the expression $\frac{1}{5}\int_0^5 L(t)\,dt$ should be obtained using a graphing calculator.

In part (c) students were asked to find, with justification, the time t in the interval $0 \le t \le 8$ when the population of fish in the lake is greatest. The key understanding here is that the rate of change of the number of fish in the lake, in number of fish per hour, is given by the difference $E(t) - L(t).$ Analysis of this difference using a graphing calculator shows that, for $0 \le t \le 8,$ the difference has exactly one sign change, occurring at $t = 6.20356.$ Before this time, $E(t) - L(t) > 0,$ so the number of fish in the lake is increasing; after this time, $E(t) - L(t) < 0,$ so the number of fish in the lake is decreasing. Thus the number of fish in the lake is greatest at $t = 6.204$ (or 6.203). An alternative justification uses the definite integral of $E(t) - L(t)$ over an interval starting at $t = 0$ to find the net change in the number of fish in the lake from time $t = 0.$ The candidates for when the fish population is greatest are the endpoints of the time interval $0 \le t \le 8$ and the one time when $E(t) - L(t) = 0,$ namely $t = 6.20356.$ Numerical evaluation of the appropriate definite integrals on a graphing calculator shows that the number of fish in the lake is greatest at $t = 6.204$ (or 6.203).

In part (d) students were asked whether the rate of change in the number of fish in the lake is increasing or decreasing at time $t = 5.$ A response should again demonstrate the understanding that the rate of change of the number of fish in the lake is given by the difference $E(t) - L(t),$ and whether this rate is increasing or decreasing at time $t = 5$ can be determined by the sign of the derivative of the difference at that time. Using a graphing calculator to find that $E'(5) - L'(5) < 0$ leads to the conclusion that the rate of change in the number of fish in the lake is decreasing at time $t = 5.$

Question 1 (continued)

For part (a) see LO CHA-4.E/EK CHA-4.E.1, LO LIM-5.A/EK LIM-5.A.3. For part (b) see LO CHA-4.B/EK CHA-4.B.1. For part (c) see LO FUN-4.B/EK FUN-4.B.1. For part (d) see LO CHA-3.C/EK CHA-3.C.1, LO CHA-2.D/EK CHA-2.D.2. This problem incorporates all four Mathematical Practices: Practice 1: Implementing Mathematical Processes, Practice 2: Connecting Representations, Practice 3: Justification, and Practice 4: Communication and Notation.

Sample: 1A
Score: 9

The response earned 9 points: 2 points in part (a), 2 points in part (b), 3 points in part (c), and 2 points in part (d). In part (a) the first point was earned with the definite integral $\int_0^5 E(t)\,dt,$ and the second point was earned with the answer $153.$ In part (b) the first point was earned with the definite integral $\int_0^5 L(t)\,dt.$ The second point was earned with multiplying the integral by $\frac{1}{5}$ and with the answer 6.059 that is accurate to three decimal places. In part (c) the first point was earned with the equation $E(t) - L(t) = 0$ in line 1 on the left. The sentence "[a]t time $t = 6.204,$ the greatest number of fish in the 8 hour period are in the lake" in lines 3 and 4 would have earned the second point without additional information. The second point was earned with the restatement "so the number of fish in the lake is greatest at $t = 6.204$ hours" in lines 7 and 8. The third point was earned with the statements "because $E(t) - L(t)$ is positive from $t = 0$ to $t = 6.204$" and "$E(t) - L(t)$ is negative from $t = 6.204$ to $t = 8$" in lines 4, 5, and 6. In part (d) the response earned the first point in line 1 with $16 + 15\sin\left(\frac{\pi t}{6}\right) - 2^{0.1t^2},$ which is equivalent to $E(t) - L(t),$ and $\frac{d}{dt}\left(16 + 15\sin\left(\frac{\pi t}{6}\right) - 2^{0.1t^2}\right)\Big|_{t=5},$ which is equivalent to $E'(5) - L'(5).$ The second point was earned with "decreasing" and the explanation, "[s]ince the derivative of $E(t) - L(t)$ at $t = 5$ is negative" in the concluding sentence.

Sample: 1B
Score: 6

The response earned 6 points: 2 points in part (a), 2 points in part (b), 2 points in part (c), and no points in part (d). In part (a) the first point was earned with the definite integral $\int_0^5 E(t)\,dt,$ and the second point was earned with the answer $153.$ In part (b) the first point was earned with the definite integral $\int_0^5 L(t)\,dt.$ The second point was earned with multiplying the integral by $\frac{1}{5}$ and with the answer 6.059 that is accurate to three decimal places. In part (c) the first point was earned with the equation $E(t) - L(t) = 0$ in line 1. The second point was earned with "[t]here is the greatest number of fish in the lake at time $t = 6.204$" in lines 3 and 4. The third point was not earned because the statement "changes signs $(+)$ to $(-)$" and $t = 6.204$ in lines 1 and 2 only provides evidence that there is a relative maximum at $t = 6.204$ rather than an absolute maximum on the interval $0 \le t \le 8.$ In part (d) no points were earned because there is no mention of $E'(5)$ and $L'(5),$ and the answer of "increasing" is incorrect.

Question 1 (continued)

Sample: 1C
Score: 3

The response earned 3 points: 2 points in part (a), 1 point in part (b), no points in part (c), and no points in part (d). In part (a) the first point was earned with the definite integral $\int_0^5 \left(20 + 15\sin\left(\frac{\pi t}{6}\right)\right) dt,$ and the second point was earned with the answer $153.$ The crossed-out work is not scored. In part (b) the first point was earned with the definite integral $\int_0^5 \left(4 + 2^{0.1t^2}\right) dt.$ The second point was not earned because the integral is not multiplied by $\frac{1}{5};$ the answer is incorrect. In part (c) no points were earned because there is no equating of $E(t) - L(t)$ to $0,$ and there is no declaration of an absolute maximum value nor a justification. In part (d) no points were earned because there is no mention of $E'(5)$ and $L'(5),$ and the answer of "Increasing" is incorrect.

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Sample Student Responses and Scoring Commentary

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Question 2

(a) $\frac{1}{2}\int_0^{\sqrt{\pi}} (r(\theta))^2 \, d\theta = 3.534292$

The area of S is 3.534.

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(b) $\frac{1}{\sqrt{\pi} - 0}\int_0^{\sqrt{\pi}} r(\theta) \, d\theta = 1.579933$

The average distance from the origin to a point on the curve $r = r(\theta)$ for $0 \le \theta \le \sqrt{\pi}$ is 1.580 (or 1.579).

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(c) $\tan\theta = \frac{y}{x} = m \;\Rightarrow\; \theta = \tan^{-1} m$

$$\frac{1}{2}\int_0^{\tan^{-1} m} (r(\theta))^2 \, d\theta = \frac{1}{2}\left(\frac{1}{2}\int_0^{\sqrt{\pi}} (r(\theta))^2 \, d\theta\right)$$

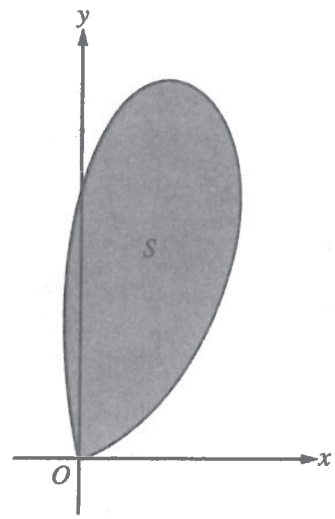
$3 : \begin{cases} 1 : \text{equates polar areas} \\ 1 : \text{inverse trigonometric function} \\ \quad \text{applied to } m \text{ as limit of} \\ \quad \text{integration} \\ 1 : \text{equation} \end{cases}$

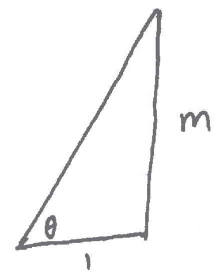
(d) As $k \to \infty$, the circle $r = k\cos\theta$ grows to enclose all points to the right of the y-axis.

$$\lim_{k\to\infty} A(k) = \frac{1}{2}\int_0^{\pi/2} (r(\theta))^2 \, d\theta$$
$$= \frac{1}{2}\int_0^{\pi/2} \left(3\sqrt{\theta}\sin\left(\theta^2\right)\right)^2 \, d\theta = 3.324$$

$2 : \begin{cases} 1 : \text{limits of integration} \\ 1 : \text{answer with integral} \end{cases}$

2A
1 of 2





2. Let S be the region bounded by the graph of the polar curve $r(\theta) = 3\sqrt{\theta}\sin\left(\theta^2\right)$ for $0 \le \theta \le \sqrt{\pi}$, as shown in the figure above.

(a) Find the area of S.

$$A_S = \frac{1}{2}\int_0^{\sqrt{\pi}} \left(3\sqrt{\theta}\sin(\theta^2)\right)^2 d\theta = \boxed{3.534}$$

(b) What is the average distance from the origin to a point on the polar curve $r(\theta) = 3\sqrt{\theta}\sin\left(\theta^2\right)$ for $0 \le \theta \le \sqrt{\pi}$?

$r(\theta)$ gives the distance from the origin to a point on the curve. Therefore, the average distance on $0 \le \theta \le \sqrt{\pi}$ is

$$\frac{\int_0^{\sqrt{\pi}} 3\sqrt{\theta}\sin(\theta^2)\,d\theta}{\sqrt{\pi} - 0} = \boxed{1.580}$$

Continue question 2 on page 7.

2A
2 of 2

(c) There is a line through the origin with positive slope m that divides the region S into two regions with equal areas. Write, but do not solve, an equation involving one or more integrals whose solution gives the value of m.

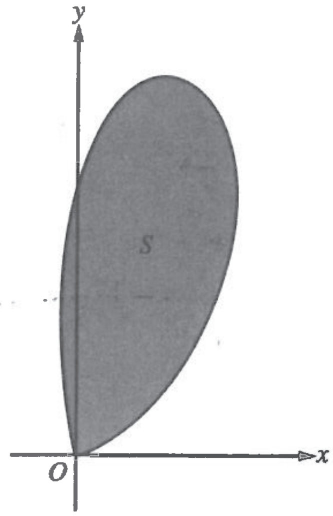
$$\frac{1}{2}\int_0^{\arctan(m)} \left(3\sqrt{\theta}\sin(\theta^2)\right)^2 d\theta = \frac{1}{2}\int_{\arctan(m)}^{\sqrt{\pi}} \left(3\sqrt{\theta}\sin(\theta^2)\right)^2 d\theta$$

(d) For $k > 0$, let $A(k)$ be the area of the portion of region S that is also inside the circle $r = k\cos\theta$. Find $\lim_{k\to\infty} A(k)$.

$$\lim_{k\to\infty} A(k) = \frac{1}{2}\int_0^{\pi/2} \left(3\sqrt{\theta}\sin(\theta^2)\right)^2 d\theta = \boxed{3.324}$$

As k approaches ∞, the circle $r = k\cos\theta$ will cover an increasingly large portion of quadrants I and IV, but because $\cos\theta < 0$ on $(\frac{\pi}{2}, \frac{3\pi}{2})$ and $\cos\theta > 0$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$, the circle will never cover the portion of S where $\frac{\pi}{2} < \theta < \sqrt{\pi}$.

2B
1 of 2



2. Let S be the region bounded by the graph of the polar curve $r(\theta) = 3\sqrt{\theta}\sin\left(\theta^2\right)$ for $0 \le \theta \le \sqrt{\pi}$, as shown in the figure above.

(a) Find the area of S.

$$\frac{1}{2}\int_0^{\sqrt{\pi}} (r(\theta))^2 d\theta = \boxed{3.534}$$

$$\frac{1}{2}\int_0^{\sqrt{\pi}} (3\sqrt{\theta}\sin(\theta^2))^2 d\theta$$

(b) What is the average distance from the origin to a point on the polar curve $r(\theta) = 3\sqrt{\theta}\sin\left(\theta^2\right)$ for $0 \le \theta \le \sqrt{\pi}$?

average r value

$$\frac{1}{\sqrt{\pi}-0}\int_0^{\sqrt{\pi}} r(\theta)\,d\theta = \boxed{1.580}$$

$$\frac{1}{\sqrt{\pi}}\int_0^{\sqrt{\pi}} (3\sqrt{\theta}\sin(\theta^2))\,d\theta$$

Continue question 2 on page 7.

2B
2of2

(c) There is a line through the origin with positive slope m that divides the region S into two regions with equal areas. Write, but do not solve, an equation involving one or more integrals whose solution gives the value of m.

$$\frac{1}{2}\int_0^m (r(\theta))\,d\theta = \frac{1}{2}\int_m^{\sqrt{\pi}} (r(\theta))\,d\theta$$

$$\frac{1}{2}\int_m^m \left(3\sqrt{\theta}\sin(\theta^2)\right)d\theta = \frac{1}{2}\int_m^{\sqrt{\pi}} \left(3\sqrt{\theta}\sin(\theta^2)\right)d\theta$$

(d) For $k > 0$, let $A(k)$ be the area of the portion of region S that is also inside the circle $r = k\cos\theta$. Find $\lim_{k\to\infty} A(k)$.

$$\boxed{\lim_{k\to\infty} A(k) = \frac{1}{2}\int_0^{\frac{\pi}{2}} (r(\theta))^2\,d\theta = 3.324}$$

As $k\to\infty$, the circle $r=k\cos\theta$ grows to encompass greater and greater proportions of the first and fourth quadrants. As $k\to\infty$ $r=k\cos\theta$ encompasses all of S that exists in the first quadrant, which can be represented as $\frac{1}{2}\int_0^{\frac{\pi}{2}} (r(\theta))^2\,d\theta$. This is because circles represented by $r=k\cos\theta$ when $r>0$ start at the origin and expand from there.

GO ON TO THE NEXT PAGE.

2C
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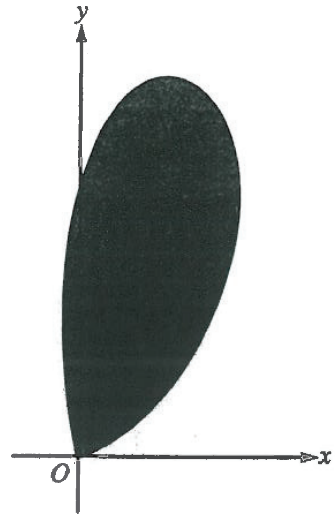


2. Let S be the region bounded by the graph of the polar curve $r(\theta) = 3\sqrt{\theta}\sin\left(\theta^2\right)$ for $0 \le \theta \le \sqrt{\pi}$, as shown in the figure above.

(a) Find the area of S.

$$A = \frac{1}{2}\int_0^{\sqrt{\pi}} \left(3\sqrt{\theta}\sin(\theta^2)\right)^2 d\theta = \boxed{3.534}$$

(b) What is the average distance from the origin to a point on the polar curve $r(\theta) = 3\sqrt{\theta}\sin\left(\theta^2\right)$ for $0 \le \theta \le \sqrt{\pi}$?

$$\text{avg} = \frac{1}{\sqrt{\pi}}\int \left(3\sqrt{\theta}\sin(\theta^2)\right)^2 d\theta =$$

Continue question 2 on page 7.

2C
2of2

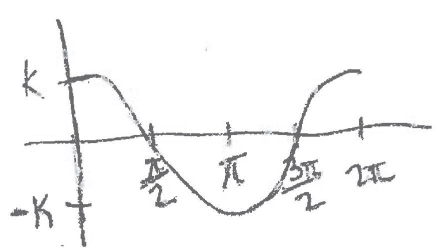
(c) There is a line through the origin with positive slope m that divides the region S into two regions with equal areas. Write, but do not solve, an equation involving one or more integrals whose solution gives the value of m.

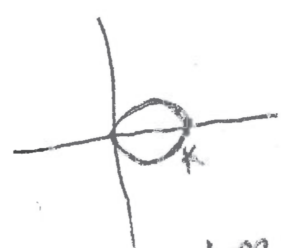
$$\frac{1}{2}\int_0^m \left(3\sqrt{\theta}\sin(\theta^2)\right)^2 d\theta = \frac{1}{2}\int_m^{\sqrt{\pi}} \left(3\sqrt{\theta}\sin(\theta^2)\right)^2 d\theta$$

(d) For $k > 0$, let $A(k)$ be the area of the portion of region S that is also inside the circle $r = k\cos\theta$. Find $\lim_{k\to\infty} A(k)$.

$3\sqrt{\theta}\sin(\theta^2) = k\cos\theta$

$r = k\cos\theta$





Area = parts that overlap

$\lim_{k\to\infty} A(k) \approx$ part of S that is in quadrant 1 $\approx \boxed{3.534}$

Question 2

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

In this problem a region S is shown in an accompanying figure, and S is identified as the region enclosed by the graph of the polar curve $r(\theta) = 3\sqrt{\theta}\sin\left(\theta^2\right)$ for $0 \le \theta \le \sqrt{\pi}$.
In part (a) students were asked to find the area of S. A response should demonstrate knowledge of the form of the integral that gives the area of a simple polar region and evaluate $\frac{1}{2}\int_0^{\sqrt{\pi}} (r(\theta))^2\, d\theta$ using the numerical integration capability of a graphing calculator.

In part (b) students were asked for the average distance from the origin to a point on the polar curve $r = r(\theta)$ for $0 \le \theta \le \sqrt{\pi}$. A response should observe that the distance from the origin to a point on the polar curve is given simply by $r(\theta)$ and then should demonstrate that the average value of $r(\theta)$ for $0 \le \theta \le \sqrt{\pi}$ is given by dividing the definite integral of $r(\theta)$ across the interval by the width of the interval. The resulting integral expression should be evaluated using the numerical integration capability of a graphing calculator.

In part (c) m denotes the positive slope of a line through the origin that divides the region S into two regions of equal areas. Students were asked to write an equation involving one or more integrals whose solution gives the value of m. A response should express the polar angle θ formed by the line and the polar axis in terms of m (namely, $\theta = \tan^{-1} m$) and use this as an upper limit in an integral that corresponds to polar area within an equation satisfying the given requirements.

In part (d) it is given that $A(k)$ represents the area of the portion of region S that is inside the circle $r = k\cos\theta$, and students were asked for the value of $\lim_{k\to\infty} A(k)$. A response should observe that any point to the right of the y-axis will eventually be inside the circle $r = k\cos\theta$ for k sufficiently large. Thus $\lim_{k\to\infty} A(k)$ is the area of the portion of S inside the first quadrant, computed as $\frac{1}{2}\int_0^{\frac{\pi}{2}} (r(\theta))^2\, d\theta$. The resulting integral expression should be evaluated using the numerical integration capability of a graphing calculator.

For part (a) see LO CHA-5.D/EK CHA-5.D.2, LO LIM-5.A/EK LIM-5.A.3. For part (b) see LO CHA-4.B/EK CHA-4.B.1, LO LIM-5.A/EK LIM-5.A.3. For part (c) see LO CHA-5.D/EK CHA-5.D.1. For part (d) see LO CHA-5.D/EK CHA-5.D.2, LO LIM-5.A/EK LIM-5.A.3. This problem incorporates the following Mathematical Practices: Practice 1: Implementing Mathematical Processes and Practice 4: Communication and Notation.

Sample: 2A
Score: 9

The response earned 9 points: 2 points in part (a), 2 points in part (b), 3 points in part (c), and 2 points in part (d). In part (a) the response earned the first point for the integral $\int_0^{\sqrt{\pi}} \left(3\sqrt{\theta}\sin\left(\theta^2\right)\right)^2 d\theta$. The factor of $\frac{1}{2}$ is not part of this point. The second point was earned for the boxed answer 3.534. In part (b) the response earned the first point for the integral $\int_0^{\sqrt{\pi}} 3\sqrt{\theta}\sin\left(\theta^2\right) d\theta$ in line 3. The denominator $\sqrt{\pi} - 0$ is not part of this point. The second point was earned for the answer 1.580 in line 3. In part (c) the response earned the first and third points for the

Question 2 (continued)

equation $\frac{1}{2}\int_0^{\arctan(m)}\left(3\sqrt{\theta}\sin\left(\theta^2\right)\right)^2\,d\theta = \frac{1}{2}\int_{\arctan(m)}^{\sqrt{\pi}}\left(3\sqrt{\theta}\sin\left(\theta^2\right)\right)^2\,d\theta.$ The first point is for equating polar areas; in this case, the area of the region from 0 to $\arctan(m)$ is equal to the area of the region from $\arctan(m)$ to $\sqrt{\pi}$. The third point is for a correct equation. The second point was earned for the limit $\arctan(m)$ in the definite integrals. In part (d) the response earned the first point for the limits of 0 and $\frac{\pi}{2}$ on the definite integral $\frac{1}{2}\int_0^{\frac{\pi}{2}}\left(3\sqrt{\theta}\sin\left(\theta^2\right)\right)^2\,d\theta$ in line 1. The second point was earned for the answer 3.324 in line 1 in the presence of the definite integral $\frac{1}{2}\int_0^{\frac{\pi}{2}}\left(3\sqrt{\theta}\sin\left(\theta^2\right)\right)^2\,d\theta.$ The commentary that is in lines 2–5 is correct but not required to earn any points.

Sample: 2B
Score: 6

The response earned 6 points: 2 points in part (a), 2 points in part (b), no points in part (c), and 2 points in part (d). In part (a) the response earned the first point for the integral $\int_0^{\sqrt{\pi}}(r(\theta))^2\,d\theta$ in line 1. The factor of $\frac{1}{2}$ is not part of this point. The second point was earned for the answer 3.534 in line 1. The definite integral $\frac{1}{2}\int_0^{\sqrt{\pi}}\left(3\sqrt{\theta}\sin\left(\theta^2\right)\right)^2\,d\theta$ in line 2 is a correct restatement of line 1. In part (b) the response earned the first point for the integral $\int_0^{\sqrt{\pi}} r(\theta)\,d\theta$ in line 2. The factor of $\frac{1}{\sqrt{\pi}-0}$ is not part of this point. The second point was earned for the answer 1.580 in line 2. The definite integral $\frac{1}{\sqrt{\pi}}\int_0^{\sqrt{\pi}}\left(3\sqrt{\theta}\sin\left(\theta^2\right)\right)d\theta$ in line 3 is a correct restatement of line 2. In part (c) the response did not earn the first point because the definite integrals $\frac{1}{2}\int_0^m(r(\theta))\,d\theta = \frac{1}{2}\int_m^{\sqrt{\pi}}(r(\theta))\,d\theta$ in line 1 and $\frac{1}{2}\int_0^m\left(3\sqrt{\theta}\sin\left(\theta^2\right)\right)d\theta = \frac{1}{2}\int_m^{\sqrt{\pi}}\left(3\sqrt{\theta}\sin\left(\theta^2\right)\right)d\theta$ in line 2 do not represent polar area because the expression for $r(\theta)$ is not squared. The second point was not earned because the limit of integration does not involve an inverse trigonometric function applied to m. The response is not eligible for the third point because the third point requires that both the first and second points have been earned. In part (d) the response earned the first point for the limits of 0 and $\frac{\pi}{2}$ on the definite integral $\frac{1}{2}\int_0^{\frac{\pi}{2}}(r(\theta))^2\,d\theta.$ The second point was earned for the answer 3.324 in line 1 in the presence of the definite integral $\frac{1}{2}\int_0^{\frac{\pi}{2}}(r(\theta))^2\,d\theta.$ The commentary below the boxed work is correct but not required to earn any points.

Sample: 2C
Score: 3

The response earned 3 points: 2 points in part (a), no points in part (b), 1 point in part (c), and no points in part (d). In part (a) the response earned the first point for the integral $\int_0^{\sqrt{\pi}}\left(3\sqrt{\theta}\sin\left(\theta^2\right)\right)^2\,d\theta.$ The factor of $\frac{1}{2}$ is not

Question 2 (continued)

part of this point. The second point was earned for the correct answer 3.534. In part (b) the response did not earn the first point because an indefinite integral $\frac{1}{\sqrt{\pi}}\int\left(3\sqrt{\theta}\sin\left(\theta^2\right)\right)^2\,d\theta$ is presented instead of a definite integral; additionally, $\left(3\sqrt{\theta}\sin\left(\theta^2\right)\right)^2$ appears as the integrand instead of $3\sqrt{\theta}\sin\left(\theta^2\right)$. The second point was not earned because no numerical answer is presented. In part (c) the response earned the first point for equating polar areas with $\frac{1}{2}\int_0^m\left(3\sqrt{\theta}\sin\left(\theta^2\right)\right)^2\,d\theta = \frac{1}{2}\int_m^{\sqrt{\pi}}\left(3\sqrt{\theta}\sin\left(\theta^2\right)\right)^2\,d\theta$; in this case, the area of the region from 0 to m is equal to the area of the region from m to $\sqrt{\pi}$. The second point was not earned because the limit of integration does not involve an inverse trigonometric function applied to m. The response is not eligible for the third point because the third point requires that both the first and second points have been earned. In part (d) the first point was not earned because no definite integral is presented with limits 0 and $\frac{\pi}{2}$. The second point was not earned because the answer 3.534 in line 4 on the right is incorrect and is not in the presence of a definite integral.

2019

AP® CollegeBoard

AP® Calculus BC

Sample Student Responses and Scoring Commentary

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Free Response Question 3

☑ Scoring Guideline

☑ Student Samples

☑ Scoring Commentary

Question 3

(a) $\int_{-6}^{5} f(x)\,dx = \int_{-6}^{-2} f(x)\,dx + \int_{-2}^{5} f(x)\,dx$

$\Rightarrow 7 = \int_{-6}^{-2} f(x)\,dx + 2 + \left(9 - \frac{9\pi}{4}\right)$

$\Rightarrow \int_{-6}^{-2} f(x)\,dx = 7 - \left(11 - \frac{9\pi}{4}\right) = \frac{9\pi}{4} - 4$

$3 : \begin{cases} 1 : \int_{-6}^{5} f(x)\,dx = \int_{-6}^{-2} f(x)\,dx + \int_{-2}^{5} f(x)\,dx \\ 1 : \int_{-2}^{5} f(x)\,dx \\ 1 : \text{answer} \end{cases}$

(b) $\int_{3}^{5} (2f'(x) + 4)\,dx = 2\int_{3}^{5} f'(x)\,dx + \int_{3}^{5} 4\,dx$

$= 2(f(5) - f(3)) + 4(5 - 3)$

$= 2\left(0 - \left(3 - \sqrt{5}\right)\right) + 8$

$= 2\left(-3 + \sqrt{5}\right) + 8 = 2 + 2\sqrt{5}$

— OR —

$\int_{3}^{5} (2f'(x) + 4)\,dx = [2f(x) + 4x]_{x=3}^{x=5}$

$= (2f(5) + 20) - (2f(3) + 12)$

$= (2 \cdot 0 + 20) - \left(2\left(3 - \sqrt{5}\right) + 12\right)$

$= 2 + 2\sqrt{5}$

$2 : \begin{cases} 1 : \text{Fundamental Theorem of Calculus} \\ 1 : \text{answer} \end{cases}$

(c) $g'(x) = f(x) = 0 \Rightarrow x = -1,\ x = \frac{1}{2},\ x = 5$

x	$g(x)$
-2	0
-1	$\frac{1}{2}$
$\frac{1}{2}$	$-\frac{1}{4}$
5	$11 - \frac{9\pi}{4}$

On the interval $-2 \le x \le 5$, the absolute maximum value of g is $g(5) = 11 - \frac{9\pi}{4}$.

$3 : \begin{cases} 1 : g'(x) = f(x) \\ 1 : \text{identifies } x = -1 \text{ as a candidate} \\ 1 : \text{answer with justification} \end{cases}$

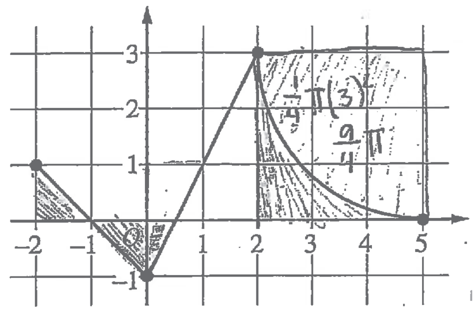
(d) $\lim_{x \to 1} \frac{10^x - 3f'(x)}{f(x) - \arctan x} = \frac{10^1 - 3f'(1)}{f(1) - \arctan 1}$

$= \frac{10 - 3 \cdot 2}{1 - \arctan 1} = \frac{4}{1 - \frac{\pi}{4}}$

1 : answer

NO CALCULATOR ALLOWED

3A 1of2



$3 \cdot \frac{3}{2} = \frac{9}{2} \cdot \frac{1}{2} = \frac{9}{4}$

Graph of f

3. The continuous function f is defined on the closed interval $-6 \le x \le 5$. The figure above shows a portion of the graph of f, consisting of two line segments and a quarter of a circle centered at the point $(5, 3)$. It is known that the point $(3, 3 - \sqrt{5})$ is on the graph of f.

(a) If $\int_{-6}^{5} f(x)\,dx = 7$, find the value of $\int_{-6}^{-2} f(x)\,dx$. Show the work that leads to your answer.

$$\int_{-6}^{-2} f(x)\,dx = \int_{-6}^{5} f(x)\,dx - \int_{-2}^{5} f(x)\,dx$$

$$\int_{-6}^{-2} f(x)\,dx = 7 - \left[\frac{1}{2} - \frac{1}{2} - \frac{1}{4} + \frac{9}{4} + \left(9 - \frac{9\pi}{4}\right)\right]$$

(b) Evaluate $\int_{3}^{5} (2f'(x) + 4)\,dx$.

$$\int_{3}^{5} 2f'(x) + 4\,dx = \int_{3}^{5} 2(f'(x) + 2)\,dx$$

$$= 2\int_{3}^{5} f'(x) + 2\,dx$$

$$= 2 \cdot \left[f(x) + 2x\right]_{3}^{5}$$

$$= 2 \cdot \left[\left[f(5) + 2(5)\right] - \left[f(3) + 2(3)\right]\right]$$

$$= 2 \cdot \left[(0 + 10) - \left((3 - \sqrt{5}) + 6\right)\right]$$

Continue question 3 on page 15

(c) The function g is given by $g(x) = \int_{-2}^{x} f(t)\,dt$. Find the absolute maximum value of g on the interval $-2 \le x \le 5$. Justify your answer.

$g'(x) = f(x)$

$g'(x) = f(x) = 0$

$x = -1,\ x = \frac{1}{2},\ x = 5$

Since $f(x)$ is continuous on $-2 \le x \le 5$, then $g(x) = \int_{-2}^{x} f(t)\,dt$ is also continuous on $-2 \le x \le 5$. Therefore, by the EVT, $g(x)$ has an absolute max. on $-2 \le x \le 5$.

Candidates for abs. max: $x = -2, -1, \frac{1}{2}, 5$

$x = -2: \int_{-2}^{-2} f(t)\,dt = 0$

$x = -1: \int_{-2}^{-1} f(t)\,dt = \frac{1}{2}$

$x = \frac{1}{2}: \int_{-2}^{1/2} f(t)\,dt = \frac{1}{2} - \frac{1}{2} - \frac{1}{4} = -\frac{1}{4}$

$x = 5: \int_{-2}^{-5} f(t)\,dt = -\frac{1}{4} + \frac{9}{4} + \left(9 - \frac{9\pi}{4}\right) = 2 + 9 - \frac{9\pi}{4} = 11 - \frac{9\pi}{4}$

The absolute maximum value of g is $11 - \frac{9\pi}{4}$ and the absolute maximum occurs at $x = 5$.

(d) Find $\lim_{x \to 1} \dfrac{10^x - 3f'(x)}{f(x) - \arctan x}$.

$\lim_{x \to 1} 10^x - 3f'(x) = 10 - 3f'(1) = 10 - 3(2) = 4$

$\lim_{x \to 1} f(x) - \arctan x = 1 - \pi/4$

$\arctan(1) = \pi/4$

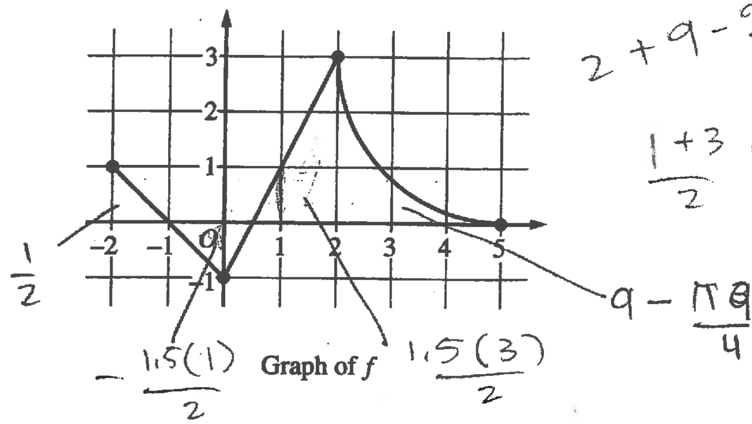
$\lim_{x \to 1} \dfrac{10^x - 3f'(x)}{f(x) - \arctan x} = \dfrac{4}{1 - \pi/4}$

NO CALCULATOR ALLOWED

3B 1of2

$2 + 9 - \frac{9\pi}{4}$

$\frac{1+3}{2}(1)$



Graph of f

3. The continuous function f is defined on the closed interval $-6 \le x \le 5$. The figure above shows a portion of the graph of f, consisting of two line segments and a quarter of a circle centered at the point $(5, 3)$. It is known that the point $(3, 3 - \sqrt{5})$ is on the graph of f.

(a) If $\int_{-6}^{5} f(x)\,dx = 7$, find the value of $\int_{-6}^{-2} f(x)\,dx$. Show the work that leads to your answer.

$$\int_{-6}^{5} f(x)dx = \int_{-6}^{-2} f(x)dx + \int_{-2}^{5} f(x)\,dx$$

$$\frac{8}{4} + 9 - \frac{9\pi}{4}$$

$$11 - \frac{9\pi}{4}$$

$$\int_{-6}^{5} f(x)dx - \int_{-2}^{5} f(x)dx = \int_{-6}^{-2} f(x)dx$$

$$7 - 11 + \frac{9\pi}{4} =$$

$$7 - \left(\frac{1}{2} - \frac{3}{4} + \frac{9}{4} + \left(9 - \frac{9\pi}{4}\right)\right) = \boxed{-2 + \frac{9\pi}{4}}$$

(b) Evaluate $\int_{3}^{5} (2f'(x) + 4)\,dx$.

$$2\int_{3}^{5} f'(x)dx + \int_{3}^{5} 4\,dx$$

$$2\left[f(x)\right]_{3}^{5} + \left[4x\right]_{3}^{5}$$

$$2(f(5) - f(3)) + 20 - 12$$

$$2(0 - 3 + \sqrt{5}) + 8$$

$$2(-3 + \sqrt{5}) + 8$$

$$-6 + 2\sqrt{5} + 8$$

$$\boxed{2 + 2\sqrt{5}}$$

Continue question 3 on page 15.

NO CALCULATOR ALLOWED

3B 2 of 2

(c) The function g is given by $g(x) = \int_{-2}^{x} f(t)\,dt$. Find the absolute maximum value of g on the interval $-2 \le x \le 5$. Justify your answer.

$+ \to -$

There are critical numbers at $x = -1, 1.5$ and 5.

$g'(x) = \frac{d}{dx} \int_{-2}^{x} f(t)dt$

$g'(x) = f(x) = 0$ at $x = -1, 1.5, 5$

$+ \quad - \quad + \quad -$; $-1 \quad 1.5 \quad 5$

$g(-1) = \int_{-2}^{-1} f(t)dt = \frac{1}{2}$

$g(5) = \int_{-2}^{5} f(t)dt = 11 - \frac{9\pi}{4}$

Since $g(5) > g(-1)$, the absolute maximum value of g is $11 - \frac{9\pi}{4}$ at $t = 5$. This value is an end point and it ~~may~~ change from positive to negative, alluding to a maximum (absolute) value. It is at a critical value

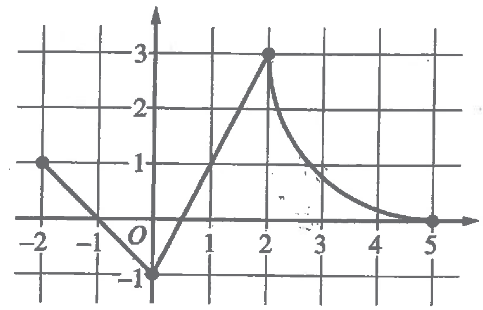
(d) Find $\lim\limits_{x \to 1} \dfrac{10^x - 3f'(x)}{f(x) - \arctan x}$.

$f'(1) = \frac{3+1}{2-0} = 2$

plug in 1 $\quad \dfrac{10^1 - 3f'(1)}{f(1) - \tan^{-1}(1)} = \dfrac{10 - 3(2)}{1 - \frac{\pi}{4}} = \dfrac{4}{\frac{\pi}{4}} = \boxed{\dfrac{1}{\pi}}$

GO ON TO THE NEXT PAGE.

3C 1of2



Graph of f

$x^2+y^2=r^2$

$x^2+y^2=3^2$

$3^2+(3-\sqrt{5})^2=9$

3. The continuous function f is defined on the closed interval $-6 \le x \le 5$. The figure above shows a portion of the graph of f, consisting of two line segments and a quarter of a circle centered at the point $(5, 3)$. It is known that the point $(3, 3-\sqrt{5})$ is on the graph of f.

(a) If $\int_{-6}^{5} f(x)\,dx = 7$, find the value of $\int_{-6}^{-2} f(x)\,dx$. Show the work that leads to your answer.

$\int_{-2}^{5} f(x)\,dx = \frac{1}{2}(1)(1) - \frac{1}{2}(1)(1) - \frac{1}{2}(1)(\frac{1}{2}) + \frac{1}{2}(\frac{3}{2})(3) +$

(b) Evaluate $\int_{3}^{5} (2f'(x)+4)\,dx$.

$\int_3^5 2f'(x)+4\ dx$

$[f(x)]^2 + 4x \Big|_3^5$

$[f(5)]^2 + 4(5) - ([f(3)]^2 + 4(3))$

$0 + 20 - (3-\sqrt{5})^2 - 12$

$8 - 14 + 6\sqrt{5} \rightarrow \boxed{-6+6\sqrt{5}}$

$(3-\sqrt{5})(3-\sqrt{5})$

$9 - 3\sqrt{5} - 3\sqrt{5} + 5$

$14 - 6\sqrt{5}$

Continue question 3 on page 15.

(c) The function g is given by $g(x) = \int_{-2}^{x} f(t)\,dt$. Find the absolute maximum value of g on the interval $-2 \le x \le 5$. Justify your answer.

$$g'(x) = \frac{d}{dx}\int_{-2}^{x} f(t)\,dt$$

$$g'(x) = f(x) = 0$$

$$x = -1,\ 0.5,\ 5$$

$+$ ∩ $-$ ∪ $+$ x $+$ $g'(x)$

-1 0.5 5

$$g(-1) = \int_{-2}^{-1} f(t)\,dt$$

(d) Find $\lim\limits_{x\to 1} \dfrac{10^x - 3f'(x)}{f(x) - \arctan x}$.

$$\rightarrow \frac{10^1 - 3f'(1)}{f(1) - \arctan(1)} = \frac{10 - 3(2)}{1 - \frac{\pi}{4}} = \frac{4}{\frac{4-\pi}{4}}$$

$$\frac{4}{1} \cdot \frac{4}{4-\pi} = \boxed{\frac{16}{4-\pi}}$$

Question 3

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

In this problem it is given that the function f is continuous on the interval $[-6, 5]$. The portion of the graph of f corresponding to $-2 \le x \le 5$ consists of two line segments and a quarter of a circle, as shown in an accompanying figure. It is noted that the point $(3, 3 - \sqrt{5})$ is on the quarter circle.

In part (a) students were asked to evaluate $\int_{-6}^{-2} f(x)\,dx$, given that $\int_{-6}^{5} f(x)\,dx = 7$. A response should demonstrate the integral property that $\int_{-6}^{-2} f(x)\,dx + \int_{-2}^{5} f(x)\,dx = \int_{-6}^{5} f(x)\,dx$ and use the interpretation of the integral in terms of the area between the graph of f and the x-axis to evaluate $\int_{-2}^{5} f(x)\,dx$ from the given graph.

In part (b) students were asked to evaluate $\int_{3}^{5} (2f'(x) + 4)\,dx$. A response should demonstrate the sum and constant multiple properties of definite integrals, together with an application of the Fundamental Theorem of Calculus that gives $\int_{3}^{5} f'(x)\,dx = f(5) - f(3)$.

In part (c) students were asked to find the absolute maximum value for the function g given by $g(x) = \int_{-2}^{x} f(t)\,dt$ on the interval $-2 \le x \le 5$. A response should demonstrate calculus techniques for optimizing a function, starting by applying the Fundamental Theorem of Calculus to obtain $g'(x) = f(x)$, and then using the supplied portion of the graph of f to find critical points for g and to evaluate g at these critical points and the endpoints of the interval.

In part (d) students were asked to evaluate $\lim_{x \to 1} \frac{10^x - 3f'(x)}{f(x) - \arctan x}$. A response should demonstrate the application of properties of limits, using the supplied portion of the graph of f to evaluate $\lim_{x \to 1} f(x)$ and $\lim_{x \to 1} f'(x)$.

For part (a) see LO FUN-6.A/EK FUN-6.A.2, LO FUN-6.A/EK FUN-6.A.1. For part (b) see LO FUN-6.B/EK FUN-6.B.2. For part (c) see LO FUN-5.A/EK FUN-5.A.2, LO FUN-4.A/EK FUN-4.A.3. For part (d) see LO LIM-1.D/EK LIM-1.D.2. This problem incorporates all four Mathematical Practices: Practice 1: Implementing Mathematical Processes, Practice 2: Connecting Representations, Practice 3: Justification, and Practice 4: Communication and Notation.

Sample: 3A
Score: 9

The response earned 9 points: 3 points in part (a), 2 points in part (b), 3 points in part (c), and 1 point in part (d). In part (a) the first point was earned with the statement of the property of definite integrals $\int_{-6}^{-2} f(x)\,dx = \int_{-6}^{5} f(x)\,dx - \int_{-2}^{5} f(x)\,dx$ in line 1. The second point was earned with $\left[\frac{1}{2} - \frac{1}{2} - \frac{1}{4} + \frac{9}{4} + \left(9 - \frac{9\pi}{4}\right)\right]$ given for $\int_{-2}^{5} f(x)\,dx$ in line 2. The third point was earned with the answer

Question 3 (continued)

$7 - \left[\frac{1}{2} - \frac{1}{2} - \frac{1}{4} + \frac{9}{4} + \left(9 - \frac{9\pi}{4}\right)\right]$ in line 2. Numerical simplification of the expression is not required. In part (b) the response earned the first point with $2 \cdot [[f(5) + 2(5)] - [f(3) + 2(3)]]$ in line 4. Note that $2 \cdot [f(x) + 2x]_3^5$ in line 3 is not sufficient to earn the first point. The second point was earned with the answer of $2 \cdot \left[(0 + 10) - \left(\left(3 - \sqrt{5}\right) + 6\right)\right]$ in the last line. Numerical simplification is not required. In part (c) the first point was earned with $g'(x) = f(x)$ in line 1 on the left. The second point was earned in line 3 on the left with $x = -1$ identified as a candidate. The second point only requires this single candidate; the other values are required for the third point. The response earned the third point by declaring the absolute maximum value of $11 - \frac{9\pi}{4}$ in line 9 and justifying this answer with the labeled values of g for both critical points and both endpoints. The statement at the top right that references the EVT (Extreme Value Theorem) is not required for the point. In part (d) the point was earned with the answer $\dfrac{4}{1 - \frac{\pi}{4}}$ in the last line. Numerical simplification is not required, and the work presented in lines 1 and 2 is not required to earn the point.

Sample: 3B
Score: 6

The response earned 6 points: 2 points in part (a), 2 points in part (b), 2 points in part (c), and no point in part (d). In part (a) the first point was earned with the statement of the property of definite integrals $\int_{-6}^{5} f(x)\,dx = \int_{-6}^{-2} f(x)\,dx + \int_{-2}^{5} f(x)\,dx$ in line 1. The second point was earned with $\left(\frac{1}{2} - \frac{3}{4} + \frac{9}{4} + \left(9 - \frac{9\pi}{4}\right)\right)$ given for $\int_{-2}^{5} f(x)\,dx$ in line 3. The third point would have been earned with the answer $7 - \left(\frac{1}{2} - \frac{3}{4} + \frac{9}{4} + \left(9 - \frac{9\pi}{4}\right)\right)$ in line 3. The numerical simplification to $-2 + \frac{9\pi}{4}$ in line 3 is incorrect, so the third point was not earned. In part (b) the response earned the first point with $2(f(5) - f(3)) + 20 - 12$ in line 3 on the left. Note that $2[f(x)]_3^5 + [4x]_3^5$ in line 2 on the left is not sufficient to earn the first point. The second point would have been earned for $2\left(0 - 3 + \sqrt{5}\right) + 8$ in line 4 on the left. Numerical simplification is not required. The boxed answer $2 + 2\sqrt{5}$ is correct, so the second point was earned. In part (c) the first point was earned with $g'(x) = f(x)$ in line 2 on the left. The inclusion of "$= 0$" is not required to earn the point. The second point was earned in line 2 on the left with $x = -1$ identified as a candidate. The second point only requires this single candidate; the other values presented are not considered for the second point. The absolute maximum value of $11 - \frac{9\pi}{4}$ is identified in line 4 on the left and declared as the absolute maximum in lines 5 and 6. The third point was not earned because of an insufficient justification. An incorrect critical point is declared at $x = 1.5,$ and the sign chart presented without explanation is not a sufficient justification for elimination of the second critical point. In part (d) the point would have been earned with the answer $\dfrac{10 - 3(2)}{1 - \frac{\pi}{4}}$. Numerical simplification is not required, though the result of $\frac{1}{\pi}$ is incorrect. The point was not earned.

Question 3 (continued)

Sample: 3C
Score: 3

The response earned 3 points: no points in part (a), no points in part (b), 2 points in part (c), and 1 point in part (d). In part (a) the property of definite integrals that is required is not stated, so the first point was not earned. Although the response begins to calculate $\int_{-2}^{5} f(x)\,dx,$ the work is incomplete, and the second point was not earned. The response is not eligible for the third point. In part (b) the antiderivative of $2f'(x)$ is reported incorrectly as $[f(x)]^2$ in line 2 on the left. The Fundamental Theorem of Calculus is not applied correctly, so the first point was not earned. Because the use of the Fundamental Theorem of Calculus is incorrect, the response is not eligible for the second point. In part (c) the first point was earned in line 2 with $g'(x) = f(x).$ The inclusion of "$= 0$" in line 2 is not required to earn the point. The second point was earned in line 3 with $x = -1$ identified as a candidate. The second point only requires this single candidate; the other values presented are not considered for the second point. An absolute maximum value of g is not given, so the third point was not earned. Note that the sign chart without explanation is not a sufficient justification. In part (d) the point would have been earned with the answer $\dfrac{10 - 3(2)}{1 - \frac{\pi}{4}}$ in line 1. Numerical simplification is not required, though the boxed result of $\dfrac{16}{4 - \pi}$ is correct and earned the point.

2019

AP CollegeBoard

AP® Calculus BC

Sample Student Responses and Scoring Commentary

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Question 4

(a) $V = \pi r^2 h = \pi(1)^2 h = \pi h$

$$\left.\frac{dV}{dt}\right|_{h=4} = \pi \left.\frac{dh}{dt}\right|_{h=4} = \pi\left(-\frac{1}{10}\sqrt{4}\right) = -\frac{\pi}{5} \text{ cubic feet per second}$$

$2: \begin{cases} 1: \dfrac{dV}{dt} = \pi\dfrac{dh}{dt} \\ 1: \text{answer with units} \end{cases}$

(b) $\dfrac{d^2h}{dt^2} = -\dfrac{1}{20\sqrt{h}} \cdot \dfrac{dh}{dt} = -\dfrac{1}{20\sqrt{h}} \cdot \left(-\dfrac{1}{10}\sqrt{h}\right) = \dfrac{1}{200}$

Because $\dfrac{d^2h}{dt^2} = \dfrac{1}{200} > 0$ for $h > 0$, the rate of change of the height is increasing when the height of the water is 3 feet.

$3: \begin{cases} 1: \dfrac{d}{dh}\left(-\dfrac{1}{10}\sqrt{h}\right) = -\dfrac{1}{20\sqrt{h}} \\ 1: \dfrac{d^2h}{dt^2} = -\dfrac{1}{20\sqrt{h}} \cdot \dfrac{dh}{dt} \\ 1: \text{answer with explanation} \end{cases}$

(c) $\dfrac{dh}{\sqrt{h}} = -\dfrac{1}{10}\,dt$

$\displaystyle\int \frac{dh}{\sqrt{h}} = \int -\frac{1}{10}\,dt$

$2\sqrt{h} = -\dfrac{1}{10}t + C$

$2\sqrt{5} = -\dfrac{1}{10} \cdot 0 + C \Rightarrow C = 2\sqrt{5}$

$2\sqrt{h} = -\dfrac{1}{10}t + 2\sqrt{5}$

$h(t) = \left(-\dfrac{1}{20}t + \sqrt{5}\right)^2$

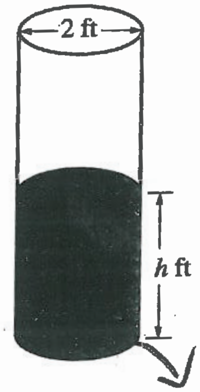
$4: \begin{cases} 1: \text{separation of variables} \\ 1: \text{antiderivatives} \\ 1: \text{constant of integration} \\ \quad \text{and uses initial condition} \\ 1: h(t) \end{cases}$

Note: $0/4$ if no separation of variables

Note: max $2/4$ [1-1-0-0] if no constant of integration

NO CALCULATOR ALLOWED

4A 1of2



4. A cylindrical barrel with a diameter of 2 feet contains collected rainwater, as shown in the figure above. The water drains out through a valve (not shown) at the bottom of the barrel. The rate of change of the height h of the water in the barrel with respect to time t is modeled by $\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$, where h is measured in feet and t is measured in seconds. (The volume V of a cylinder with radius r and height h is $V = \pi r^2 h$.)

(a) Find the rate of change of the volume of water in the barrel with respect to time when the height of the water is 4 feet. Indicate units of measure.

$h=4$

$\frac{dh}{dt} = \frac{-1}{10}\sqrt{4} = \frac{-1}{5}$

$\frac{dV}{dt} = ?$

$r = 1$

$\frac{dr}{dt} = 0$

$\frac{dV}{dt} = \pi\left[r^2\frac{dh}{dt} + h(2r)\left(\frac{dr}{dt}\right)\right]$

$\frac{dV}{dt} = \pi\left[(1)\left(\frac{-1}{5}\right) + (4)(2)(0)\right]$

$\frac{dV}{dt} = \frac{-\pi}{5}\ \text{ft}^3/\text{sec}$

Continue question 4 on page 17.

(b) When the height of the water is 3 feet, is the rate of change of the height of the water with respect to time increasing or decreasing? Explain your reasoning.

$h=3$

$\frac{d^2h}{dt^2} = ?$

$\frac{-1}{10} h^{1/2}$

$$\frac{d^2h}{dt^2} = \frac{-1}{20} h^{-1/2} \frac{dh}{dt} = \frac{-1}{20\sqrt{3}} \left(\frac{-1}{10}\sqrt{3}\right) = \frac{1}{200}$$

The rate of change of height is increasing since $\frac{d^2h}{dt^2}$ at $h=3$ is positive.

$$\frac{1}{200}$$

(c) At time $t = 0$ seconds, the height of the water is 5 feet. Use separation of variables to find an expression for h in terms of t.

$$\int h^{-1/2} dh = \int \frac{-1}{10} dt$$

$$2h^{1/2} = \frac{-1}{10} t + C,$$

$$h^{1/2} = \frac{-1}{20} t + C$$

$$h = \left(\frac{-1}{20} t + C\right)^2$$

$$5 = \left(\frac{-1}{20}(0) + C\right)^2$$

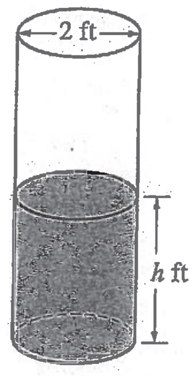
$$C = \sqrt{5}$$

$$h = \left(\frac{-1}{20} t + \sqrt{5}\right)^2$$

GO ON TO THE NEXT PAGE.

NO CALCULATOR ALLOWED

4B
1 of 2



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(a) Find the rate of change of the volume of water in the barrel with respect to time when the height of the water is 4 feet. Indicate units of measure.

$$\frac{dV}{dt} = \pi r^2 \frac{dh}{dt}$$

for $h = 4$:

$$\frac{dV}{dt} = \pi(1)^2 \cdot \left(-\frac{1}{10}\sqrt{4}\right) = \boxed{-\frac{\pi}{5} \text{ feet}^3/\text{s}}$$

Continue question 4 on page 17.

(b) When the height of the water is 3 feet, is the rate of change of the height of the water with respect to time increasing or decreasing? Explain your reasoning.

at $h=3$, $\frac{dh}{dt} = -\frac{1}{10}\sqrt{3}$, which is negative, so the amount of water is decreasing

(c) At time $t = 0$ seconds, the height of the water is 5 feet. Use separation of variables to find an expression for h in terms of t.

$\frac{1}{\sqrt{h}}dh = -\frac{1}{10}dt$

$\int \frac{1}{\sqrt{h}}dh = \int -\frac{1}{10}\,dt$

$2\sqrt{h} = -\frac{t}{10} + C$

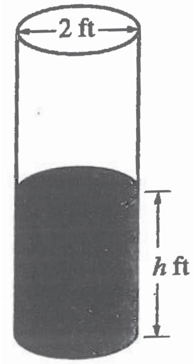
$2\sqrt{5} = -\frac{0}{10} + C$

$C = 2\sqrt{5}$

$2\sqrt{h} = -\frac{t}{10} + 2\sqrt{5}$

$\sqrt{h} = -\frac{t}{20} + \sqrt{5}$

$\boxed{h = \left(-\frac{t}{20} + \sqrt{5}\right)^2}$



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(a) Find the rate of change of the volume of water in the barrel with respect to time when the height of the water is 4 feet. Indicate units of measure.

$\frac{dv}{dt} = ?$ $h=4$

$V=\pi r^2 h$

$r=1$

$\frac{dh}{dt} = \frac{-1}{10}\sqrt{h}$

$V=\pi r^2 h$

$\frac{dV}{dt} = 2\pi r \frac{dh}{dt}$

$\frac{dv}{dt} = 2\pi(1)\left(-\frac{1}{10}\sqrt{4}\right)$

$\frac{dv}{dt} = 2\pi \cdot -\frac{2}{10} = -\frac{2\pi}{5}$ ft^3 persecond

Continue question 4 on page 17.

(b) When the height of the water is 3 feet, is the rate of change of the height of the water with respect to time increasing or decreasing? Explain your reasoning.

$\frac{dh}{dt}$

$r(t) = \frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$

$r'(t) = -\frac{1}{10} \cdot \frac{1}{2} h^{-\frac{1}{2}}$

$r'(3) = -\frac{1}{10} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{3}} = -\frac{1}{20\sqrt{3}} < 0$

When the height of the water is 3 feet the rate of change of height of the water is decreasing because $r'(t) < 0$.

(c) At time $t = 0$ seconds, the height of the water is 5 feet. Use separation of variables to find an expression for h in terms of t.

$\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$

$\frac{1}{\sqrt{h}}\, dh = -\frac{1}{10}\, dt$

$2h^{1/2} + C_1 = -\frac{1}{10} t + C_2$

$h = \sqrt{-\frac{1}{20} t + C}$

$5 = \sqrt{-\frac{1}{20}(0) + C}$

$C = 25$

$h = \sqrt{-\frac{1}{20} t + 25}$

Question 4

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

The context for this problem is a cylindrical barrel with a diameter of 2 feet that contains collected rainwater, some of which drains out through a valve in the bottom of the barrel. The rate of change of the height h of the water in the barrel with respect to time t is modeled by $\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$, where h is measured in feet, and t is measured in seconds.

In part (a) students were asked to find the rate of change of the volume of water in the barrel with respect to time when $h = 4$ feet. A response should use the geometric relationship between the volume V of water in the barrel and height h and incorporate the given expression for $\frac{dh}{dt}$.

In part (b) students were asked to determine whether the rate of change of the height of water in the barrel is increasing or decreasing when $h = 3$ feet. A response should demonstrate facility with the chain rule to differentiate $\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$ with respect to time to obtain $\frac{d^2h}{dt^2} = -\frac{1}{20\sqrt{h}} \cdot \frac{dh}{dt} = -\frac{1}{20\sqrt{h}} \cdot \left(-\frac{1}{10}\sqrt{h}\right) = \frac{1}{200}$. Because $\frac{d^2h}{dt^2} > 0$, a response should conclude that the rate of change of the height of the water in the barrel is increasing.

In part (c) students were given that the height of the water is 5 feet at time $t = 0$ and then asked to use the technique of separation of variables to find an expression for h in terms of t. A response should demonstrate the application of separation of variables to solve the differential equation $\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$ for h and then incorporate the initial condition that $h(0) = 5$ to find the particular solution $h(t)$ to the differential equation.

For part (a) see LO CHA-3.D/EK CHA-3.D.1, LO CHA-3.E/EK CHA-3.E.1. For part (b) see LO FUN-4.E/EK FUN-4.E.2. For part (c) see LO FUN-7.D/EK FUN-7.D.1, LO FUN-6.C/EK FUN-6.C.2, LO FUN-7.E/EK FUN-7.E.1. This problem incorporates all four Mathematical Practices: Practice 1: Implementing Mathematical Processes, Practice 2: Connecting Representations, Practice 3: Justification, and Practice 4: Communication and Notation.

Sample: 4A
Score: 9

The response earned 9 points: 2 points in part (a), 3 points in part (b), and 4 points in part (c). In part (a) the response earned the first point with the presentation of a correct expression for the derivative of V with respect to t, $\frac{dV}{dt} = \pi\left[r^2\frac{dh}{dt} + h(2r)\left(\frac{dr}{dt}\right)\right]$, in line 1 on the right. The second point would have been earned with $\frac{dV}{dt} = \pi\left[(1)\left(\frac{-1}{5}\right) + (4)(2)(0)\right]$ in line 2 on the right. Although numerical simplification is not required, the response simplifies the expression in line 3 on the right and adds units to produce $-\frac{\pi}{5}$ ft^3/sec. Thus the second point was earned. In part (b) the response presents a correct second derivative of h with respect to t, $\frac{d^2h}{dt^2} = -\frac{1}{20}h^{-\frac{1}{2}}\frac{dh}{dt}$, in line 1 on the right and earned both the first and second points. The response earned the

Question 4 (continued)

third point in lines 2, 3, and 4 on the right with "[t]he rate of change of height is increasing since $\frac{d^2h}{dt^2}$ at $h = 3$ is positive." In part (c) the response earned the first point with a correct separation of variables $\int h^{-\frac{1}{2}}\, dh = \int \frac{-1}{10}\, dt$ in line 1. The correct antiderivatives, $2h^{\frac{1}{2}}$ and $-\frac{1}{10}t,$ are presented in line 2, and the response earned the second point. In lines 2 and 5, the response includes a constant of integration and uses the initial condition $h(0) = 5$ by substituting 0 for t and 5 for $h.$ The response earned the third point. The response solves for h in terms of t and earned the fourth point with $h = \left(\frac{-1}{20}t + \sqrt{5}\right)^2$ in line 7.

Sample: 4B
Score: 6

The response earned 6 points: 2 points in part (a), no points in part (b), and 4 points in part (c). In part (a) the response earned the first point with the presentation of a correct expression for the derivative of V with respect to $t,$ $\frac{dV}{dt} = \pi r^2 \frac{dh}{dt},$ while handling r as a constant. The second point would have been earned with $\frac{dV}{dt} = \pi(1)^2\left(-\frac{1}{10}\sqrt{4}\right)$ in line 3. Although numerical simplification is not required, the response simplifies the expression in line 3 and adds units to produce $-\frac{\pi}{5}$ feet^3/s. Thus the second point was earned. In part (b) the response does not include the derivative of $\frac{dh}{dt},$ so the first and second points were not earned. Because there is no second derivative, the response is not eligible for the third point. In part (c) the response earned the first point with a correct separation of variables $\frac{1}{\sqrt{h}}\, dh = -\frac{1}{10}\, dt$ in line 1 on the left. The correct antiderivatives, $2\sqrt{h}$ and $-\frac{t}{10},$ are presented in line 3 on the left, and the response earned the second point. In lines 3 and 4 on the left, the response includes a constant of integration and uses the initial condition $h(0) = 5$ by substituting 0 for t and 5 for $h.$ The response earned the third point. The response solves for h in terms of t and earned the fourth point with $h = \left(-\frac{t}{20} + \sqrt{5}\right)^2$ in the box in line 3 on the right.

Sample: 4C
Score: 3

The response earned 3 points: no points in part (a), 1 point in part (b), and 2 points in part (c). In part (a) the response presents an incorrect expression for the derivative of V with respect to $t,$ $\frac{dV}{dt} = 2\pi r \frac{dh}{dt},$ in line 3 on the left. The first point was not earned, and this error makes the response not eligible for the second point. In part (b) the response defines $r(t) = \frac{dh}{dt}$ in line 1. The response earned the first point with $r'(t) = -\frac{1}{10} \cdot \frac{1}{2} h^{-\frac{1}{2}}$ in line 2. The expression is identified as the second derivative of h with respect to $t;$ however, $r'(t)$ does not

Question 4 (continued)

include a factor of $\frac{dh}{dt}$. Thus the second point was not earned, and the response is not eligible for the third point.

In part (c) the response earned the first point with a correct separation of variables $\frac{1}{\sqrt{h}}\, dh = -\frac{1}{10}\, dt$ in line 2 on the left. The correct antiderivatives, $2h^{\frac{1}{2}}$ and $-\frac{1}{10}t$, are presented in line 3 on the left, and the response earned the second point. In line 3 on the left, the response includes constants of integration. The response incorrectly solves for h in terms of t in line 4 before using the initial condition $h(0) = 5$ in line 5. The resulting expression $h = \sqrt{-\frac{1}{20}t + C}$ in line 4 on the left is incorrect. Thus the response is not eligible for the third and fourth points.

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Sample Student Responses and Scoring Commentary

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Question 5

(a) $f'(x) = \dfrac{-(2x-2)}{\left(x^2 - 2x + k\right)^2}$

$f'(0) = \dfrac{2}{k^2} = 6 \Rightarrow k^2 = \dfrac{1}{3} \Rightarrow k = \dfrac{1}{\sqrt{3}}$

$3 : \begin{cases} 1 : \text{denominator of } f'(x) \\ 1 : f'(x) \\ 1 : \text{answer} \end{cases}$

(b) $\dfrac{1}{x^2 - 2x - 8} = \dfrac{1}{(x-4)(x+2)} = \dfrac{A}{x-4} + \dfrac{B}{x+2}$

$\Rightarrow 1 = A(x+2) + B(x-4)$

$\Rightarrow A = \dfrac{1}{6},\ B = -\dfrac{1}{6}$

$$\int_0^1 f(x)\,dx = \int_0^1 \left(\frac{\frac{1}{6}}{x-4} - \frac{\frac{1}{6}}{x+2} \right) dx$$
$$= \left[\frac{1}{6}\ln|x-4| - \frac{1}{6}\ln|x+2| \right]_{x=0}^{x=1}$$
$$= \left(\frac{1}{6}\ln 3 - \frac{1}{6}\ln 3 \right) - \left(\frac{1}{6}\ln 4 - \frac{1}{6}\ln 2 \right) = -\frac{1}{6}\ln 2$$

$3 : \begin{cases} 1 : \text{partial fraction decomposition} \\ 1 : \text{antiderivatives} \\ 1 : \text{answer} \end{cases}$

(c) $$\int_0^2 \frac{1}{x^2 - 2x + 1}\,dx = \int_0^2 \frac{1}{(x-1)^2}\,dx = \int_0^1 \frac{1}{(x-1)^2}\,dx + \int_1^2 \frac{1}{(x-1)^2}\,dx$$
$$= \lim_{b\to 1^-} \int_0^b \frac{1}{(x-1)^2}\,dx + \lim_{b\to 1^+} \int_b^2 \frac{1}{(x-1)^2}\,dx$$
$$= \lim_{b\to 1^-} \left(-\frac{1}{x-1} \Big|_{x=0}^{x=b} \right) + \lim_{b\to 1^+} \left(-\frac{1}{x-1} \Big|_{x=b}^{x=2} \right)$$
$$= \lim_{b\to 1^-} \left(-\frac{1}{b-1} - 1 \right) + \lim_{b\to 1^+} \left(-1 + \frac{1}{b-1} \right)$$

Because $\lim_{b\to 1^-} \left(-\frac{1}{b-1} \right)$ does not exist, the integral diverges.

$3 : \begin{cases} 1 : \text{improper integral} \\ 1 : \text{antiderivative} \\ 1 : \text{answer with reason} \end{cases}$

NO CALCULATOR ALLOWED

5A
1 of 2

5. Consider the family of functions $f(x) = \dfrac{1}{x^2 - 2x + k}$, where k is a constant.

(a) Find the value of k, for $k > 0$, such that the slope of the line tangent to the graph of f at $x = 0$ equals 6.

$$f(x) = (x^2-2x+k)^{-1}$$
$$f'(x) = -(x^2-2x+k)^{-2}(2x-2)$$
$$= -\frac{2x-2}{(x^2-2x+k)^2}$$
$$f'(0) = -\frac{-2}{k^2} = 6$$
$$6k^2 = 2$$
$$\boxed{k = \frac{1}{\sqrt{3}}}$$

(b) For $k = -8$, find the value of $\int_0^1 f(x)\,dx$.

$$\frac{A}{(x-4)} + \frac{B}{(x+2)} = \frac{1}{(x-4)(x+2)}$$
$$A(x+2)+B(x-4) = 1$$
$$6A = 1 \quad A+B=0$$
$$A = \frac{1}{6} \quad B = -\frac{1}{6}$$

$$\int_0^1 \frac{1}{x^2-2x-8}\,dx$$
$$\int_0^1 \frac{\frac{1}{6}}{x-4} - \frac{\frac{1}{6}}{x+2}\,dx =$$
$$\frac{1}{6}\ln|x-4| - \frac{1}{6}\ln|x+2|\Big|_0^1 =$$
$$\frac{1}{6}\ln\left|\frac{x-4}{x+2}\right|\Big|_0^1 = \frac{1}{6}\ln\left|\frac{-3}{3}\right| - \frac{1}{6}\ln\left|\frac{-4}{2}\right| = \frac{1}{6}(0-\ln 2) = \boxed{-\frac{1}{6}\ln 2}$$

Continue question 5 on page 19.

(c) For $k = 1$, find the value of $\int_0^2 f(x)\,dx$ or show that it diverges.

$$\int_0^2 \frac{1}{x^2-2x+1}\,dx$$

$x^2-2x+1=0$

$(x-1)^2=0$

$x=1$

$$\lim_{R\to1^-}\int_0^R \frac{1}{(x-1)^2}\,dx + \lim_{R\to1^+}\int_R^2 \frac{1}{(x-1)^2}\,dx$$

$$\lim_{R\to1^-} -\frac{1}{x-1}\Big|_0^R + \lim_{R\to1^+} -\frac{1}{x-1}\Big|_R^2$$

$$\lim_{R\to1^-}\left(-\frac{1}{R-1}+\frac{1}{-1}\right)+\lim_{R\to1^+}\left(-\frac{1}{2-1}+\frac{1}{R-1}\right)$$

$$\infty - 1 - 1 + \infty$$

$\boxed{\text{The integral diverges}}$

5B
1 of 2

5. Consider the family of functions $f(x) = \dfrac{1}{x^2 - 2x + k}$, where k is a constant.

(a) Find the value of k, for $k > 0$, such that the slope of the line tangent to the graph of f at $x = 0$ equals 6.

$$f'(x) = \frac{(x^2-2x+k)(0)-1(2x-2)}{(x^2-2x+k)^2}$$

$$f'(x) = \frac{-2x+2}{(x^2-2x+k)^2}$$

$$f'(0) = 6$$

$$\frac{-2(0)+2}{(0^2-2(0)+k)^2} = 6$$

$$\frac{2}{k^2} = 6$$

$$2 = 6k^2$$

$$\boxed{k = \pm\sqrt{\frac{1}{3}}}$$

(b) For $k = -8$, find the value of $\int_0^1 f(x)\,dx$.

$$\frac{1}{(x-4)(x+2)} = \frac{A}{x-4} + \frac{B}{x+2}$$

$$1 = A(x+2) + B(x-4)$$

$$1 = (A+B)x + 2A - 4B$$

$$(A+B=0)-2$$

$$2A-4B=1$$

$$-2A-2B=0 \qquad A-\frac{1}{6}=0$$

$$-6B=1 \qquad A=\frac{1}{6}$$

$$B=-\frac{1}{6} \qquad \frac{1}{(x-4)(x+2)} = \frac{\frac{1}{6}}{x-4} - \frac{\frac{1}{6}}{x+2}$$

$$\int_0^1 \frac{1}{x^2-2x-8}\,dx = \int_0^1 \frac{1}{(x-4)(x+2)}$$

$$= \frac{1}{6}\int_0^1 \frac{1}{x-4}\,dx - \frac{1}{6}\int_0^1 \frac{1}{x+2}\,dx$$

$$= \frac{1}{6}\left[\ln|x-4|\Big|_0^1 - \ln|x+2|\Big|_0^1\right]$$

$$= \frac{1}{6}\left[\ln 3 - \ln 4 - \ln 3 + \ln 2\right]$$

$$= \frac{1}{6}(\ln 2 - \ln 4)$$

$$= \boxed{\frac{1}{6}\ln\frac{1}{2}}$$

Continue question 5 on page 19.

(c) For $k = 1$, find the value of $\int_0^2 f(x)\,dx$ or show that it diverges.

$x^2 - 2x + 1 = (x-1)^2$

$\int_0^2 \frac{1}{x^2-2x+1}\,dx$

$\int_0^2 \frac{1}{(x-1)^2}\,dx$ let $u = x-1$

$\frac{du}{dx} = 1$

$du = dx$

$\int_{-1}^{1} \frac{1}{u^2}\,du$

$-\frac{1}{u}\Big|_{-1}^{1}$

$-\frac{1}{1} - \frac{1}{1}$

$\boxed{-2}$

NO CALCULATOR ALLOWED

5C
1 2

5. Consider the family of functions $f(x) = \dfrac{1}{x^2 - 2x + k}$, where k is a constant.

(a) Find the value of k, for $k > 0$, such that the slope of the line tangent to the graph of f at $x = 0$ equals 6.

$f'(x) = \dfrac{(x^2-2x+k)-(2x-2)}{(x^2-2x+k)^2} = \dfrac{x^2-4x+2+k}{(x^2-2x+k)^2}$ $\quad 0^2-4(0)+2+k=6 \quad 2+k=6$

$k=4$

(b) For $k = -8$, find the value of $\int_0^1 f(x)\,dx$.

$\int_0^1 \frac{1}{x^2-2x-8}dx = \int_0^1 \frac{1}{(x-4)(x+2)}dx = \int_0^1 \frac{A}{x-4} + \frac{B}{x+2}\,dx = \frac{1}{6}\int_0^1 \frac{1}{x-4}\,dx - \frac{1}{6}\int_0^1 \frac{1}{x+2}\,dx$

$1 = Ax + 2A + Bx - 4B$ $\quad A+B=0 \quad A=-B$

$2A-4B=1$

$-6B=1$

$B=\frac{-1}{6} \quad A=\frac{1}{6}$

$\frac{1}{6}\left(\ln\frac{-1}{3} - \ln\frac{-1}{4}\right) - \frac{1}{6}\left(\ln\frac{1}{3} - \ln\frac{1}{2}\right)$

$\frac{1}{6}\ln\frac{4}{3} - \frac{1}{6}\ln\frac{2}{3}$

$\frac{1}{6}\ln\frac{4}{2} = \frac{\ln 2}{6}$

Continue question 5 on page 19.

NO CALCULATOR ALLOWED

5C
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(c) For $k = 1$, find the value of $\int_0^2 f(x)\,dx$ or show that it diverges.

$\int_0^2 \frac{1}{x^2-2x+1}\,dx = \int_0^2 \frac{1}{(x-1)^2}\,dx = \left.\frac{-1}{x-1}\right]_0^2 = \frac{-1}{1} - 1 = -2$

GO ON TO THE NEXT PAGE.

Question 5

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

This problem deals with a family of functions $f(x) = \dfrac{1}{x^2 - 2x + k}$, where k is a constant.

In part (a) students were asked to find the positive value of k such that the slope of the line tangent to the graph of f at $x = 0$ equals 6. A response should demonstrate differentiation rules to find $f'(x)$ and then identify $f'(0)$ as the slope of the line tangent to the graph of f at $x = 0$, so the k can be found by solving $f'(0) = 6$.

In part (b) students were asked to evaluate $\int_0^1 f(x)\,dx$ in the case where $k = -8$. A response should demonstrate that, with $k = -8$, $f(x)$ can be expressed using partial fractions as $f(x) = \dfrac{1}{x^2 - 2x - 8} = \dfrac{\frac{1}{6}}{x-4} - \dfrac{\frac{1}{6}}{x+2}$. Then $\int_0^1 f(x)\,dx$ can be evaluated using antidifferentiation and the Fundamental Theorem of Calculus.

In part (c) students were asked to evaluate $\int_0^2 f(x)\,dx$ or show that it diverges, in the case where $k = 1$. A response should note that $x^2 - 2x + 1 = (x-1)^2$, so the graph of f has a vertical asymptote at $x = 1$ and $\int_0^2 f(x)\,dx$ is an improper integral. Thus $\int_0^2 f(x)\,dx$ is the sum $\int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx$, providing each of the summands converges. Expressing either summand as a one-sided limit of a proper integral, a response should demonstrate that the summand diverges and conclude that $\int_0^2 f(x)\,dx$ diverges.

For part (a) see LO FUN-3.C/EK FUN-3.C.1, LO CHA-2.C/EK CHA-2.C.1. For part (b) see LO FUN-6.F.b/EK FUN-6.F.1, LO FUN-6.C/EK FUN-6.C.2, LO FUN-6.B/EK FUN-6.B.3. For part (c) see LO LIM-6.A/EK LIM-6.A.1, LO FUN-6.C/EK FUN-6.C.2, LO LIM-6.A/EK LIM-6.A.2. This problem incorporates the following Mathematical Practices: Practice 1: Implementing Mathematical Processes, Practice 2: Connecting Representations, and Practice 4: Communication and Notation.

Sample: 5A
Score: 9

The response earned 9 points: 3 points in part (a), 3 points in part (b), and 3 points in part (c). In part (a) the response earned the first point in line 2 with the term $\left(x^2 - 2x + k\right)^{-2}$ in the presented derivative expression $f'(x) = -\left(x^2 - 2x + k\right)^{-2}(2x - 2)$. The second point was also earned in line 2 with this presented expression for the derivative $f'(x)$. The third point was earned in line 6 with the declaration that $k = \dfrac{1}{\sqrt{3}}$ and the consistent work leading to this value. In part (b) the response earned the first point in line 1 on the right with the equation $\dfrac{A}{(x-4)} + \dfrac{B}{(x+2)} = \dfrac{1}{(x-4)(x+2)}$ and the declaration in line 4 on the right that $A = \dfrac{1}{6}$ and $B = -\dfrac{1}{6}$. The

Question 5 (continued)

response earned the second point in line 3 on the left with the antiderivatives $\frac{1}{6}\ln|x-4| - \frac{1}{6}\ln|x+2|$. The response would have earned the third point in line 4 on the left with the expression $\frac{1}{6}\ln\left|\frac{-3}{3}\right| - \frac{1}{6}\ln\left|\frac{-4}{2}\right|$ and no numerical simplification. The response simplifies the expression correctly, and the third point was earned with $-\frac{1}{6}\ln 2$. In part (c) the response earned the first point in line 2 with the expression $\lim_{R\to 1^-}\int_0^R \frac{1}{(x-1)^2}\,dx + \lim_{R\to 1^+}\int_R^2 \frac{1}{(x-1)^2}\,dx$. The response earned the second point in line 3 with the antiderivative $-\frac{1}{x-1}$ in the expression $\lim_{R\to 1^-} -\frac{1}{x-1}\Big|_0^R + \lim_{R\to 1^+} -\frac{1}{x-1}\Big|_R^2$. The response earned the third point in line 6 with the boxed conclusion "[t]he integral diverges" and with the correct work that includes both one-sided limits evaluated in line 5 as $\infty - 1 - 1 + \infty$.

Sample: 5B
Score: 6

The response earned 6 points: 2 points in part (a), 3 points in part (b), and 1 point in part (c). In part (a) the response earned the first point in line 1 with the denominator $\left(x^2-2x+k\right)^2$ in the presented derivative expression $f'(x) = \frac{\left(x^2-2x+k\right)(0) - 1(2x-2)}{\left(x^2-2x+k\right)^2}$. The second point was also earned in line 1 with this presented expression for the derivative $f'(x)$. The response did not earn the third point, as the value $k = \pm\sqrt{\frac{1}{3}}$ in line 6 does not utilize the given information that $k > 0$. In part (b) the response earned the first point in line 1 on the right with the equation $\frac{1}{(x-4)(x+2)} = \frac{A}{x-4} + \frac{B}{x+2}$ and the declaration in lines 7 and 8 on the right that $A = \frac{1}{6}$ and $B = -\frac{1}{6}$. The response earned the second point in line 3 on the left with the antiderivatives $\frac{1}{6}\left[\ln|x-4|\Big|_0^1 - \ln|x+2|\Big|_0^1\right]$. The response would have earned the third point in line 4 on the left with the expression $\frac{1}{6}[\ln 3 - \ln 4 - \ln 3 + \ln 2]$ and no numerical simplification. The response simplifies this expression correctly to the boxed $\frac{1}{6}\ln\frac{1}{2}$ in line 6 on the left, and the third point was earned with this final expression. In part (c) the response did not earn the first point because there is no indication that the integral $\int_0^2 \frac{1}{(x-1)^2}\,dx$ is improper. The response earned the second point by letting $u = x - 1$ in line 2 and with antiderivative $-\frac{1}{u}$ in line 4. The response did not earn the third point because there is no conclusion of divergence.

Question 5 (continued)

Sample: 5C
Score: 3

The response earned 3 points: 1 point in part (a), 1 point in part (b), and 1 point in part (c). In part (a) the response earned the first point in line 1 with the correct denominator $\left(x^2 - 2x + k\right)^2$ in the presented derivative expression $f'(x) = \dfrac{\left(x^2 - 2x + k\right) - (2x - 2)}{\left(x^2 - 2x + k\right)^2}$. The response did not earn the second point because the presented derivative for $f'(x)$ in line 1 is incorrect. The response did not earn the third point because the circled answer $k = 4$ on the right is incorrect. In part (b) the response earned the first point in line 1 with the equation $\int_0^1 \frac{1}{(x-4)(x+2)}\,dx = \int_0^1 \frac{A}{x-4} + \frac{B}{x+2}\,dx$ and with the declaration in the last line on the right that $B = \frac{-1}{6}$ and $A = \frac{1}{6}$. The missing parentheses in the integrand on the right side of the equation do not impact earning the point. The response did not earn the second point because there is no antiderivative presented. The response did not earn the third point due to the undefined expression $\frac{1}{6}\left(\ln\frac{-1}{3} - \ln\frac{-1}{4}\right) - \frac{1}{6}\left(\ln\frac{1}{3} - \ln\frac{1}{2}\right)$ in line 3 on the left. Note that even if the circled answer $\frac{\ln 2}{6}$ had been correct, this response would not have been eligible for the third point because of the undefined, incorrect expression. In part (c) the response did not earn the first point because there is no indication that the integral $\int_0^2 \frac{1}{(x-1)^2}\,dx$ is improper. The response earned the second point with the antiderivative $\frac{-1}{x-1}$. The response did not earn the third point because there is no conclusion of divergence.

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Sample Student Responses and Scoring Commentary

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Question 6

(a) $f(0) = 3$ and $f'(0) = -2$

The third-degree Taylor polynomial for f about $x = 0$ is

$$3 - 2x + \frac{3}{2!}x^2 + \frac{-\frac{23}{2}}{3!}x^3 = 3 - 2x + \frac{3}{2}x^2 - \frac{23}{12}x^3.$$

$2 : \begin{cases} 1 : \text{two terms} \\ 1 : \text{remaining terms} \end{cases}$

(b) The first three nonzero terms of the Maclaurin series for e^x are $1 + x + \frac{1}{2!}x^2$.

The second-degree Taylor polynomial for $e^x f(x)$ about $x = 0$ is

$$3\left(1 + x + \frac{1}{2!}x^2\right) - 2x(1 + x) + \frac{3}{2}x^2(1)$$
$$= 3 + (3 - 2)x + \left(\frac{3}{2} - 2 + \frac{3}{2}\right)x^2$$
$$= 3 + x + x^2.$$

$2 : \begin{cases} 1 : \text{three terms for } e^x \\ 1 : \text{three terms for } e^x f(x) \end{cases}$

(c) $h(1) = \int_0^1 f(t)\,dt$

$$\approx \int_0^1 \left(3 - 2t + \frac{3}{2}t^2 - \frac{23}{12}t^3\right) dt$$
$$= \left[3t - t^2 + \frac{1}{2}t^3 - \frac{23}{48}t^4\right]_{t=0}^{t=1}$$
$$= 3 - 1 + \frac{1}{2} - \frac{23}{48} = \frac{97}{48}$$

$2 : \begin{cases} 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

(d) The alternating series error bound is the absolute value of the first omitted term of the series for $h(1)$.

$$\int_0^1 \left(\frac{54}{4!}t^4\right) dt = \left[\frac{9}{20}t^5\right]_{t=0}^{t=1} = \frac{9}{20}$$

$$\text{Error} \le \left|\frac{9}{20}\right| = 0.45$$

$3 : \begin{cases} 1 : \text{uses fourth-degree term of Maclaurin series for } f \\ 1 : \text{uses first omitted term of series for } h(1) \\ 1 : \text{error bound} \end{cases}$

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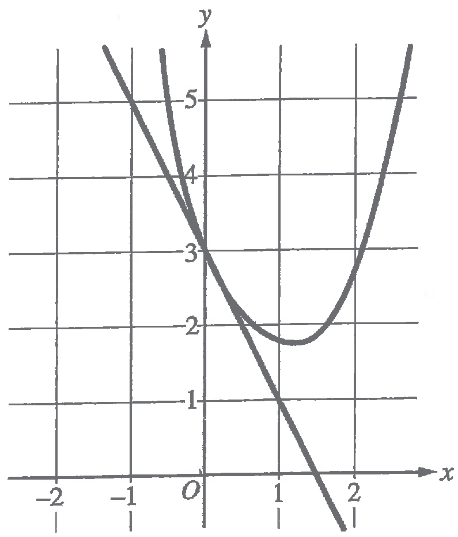

n	$f^{(n)}(0)$
2	3
3	$-\frac{23}{2}$
4	54

6. A function f has derivatives of all orders for all real numbers x. A portion of the graph of f is shown above, along with the line tangent to the graph of f at $x = 0$. Selected derivatives of f at $x = 0$ are given in the table above.

(a) Write the third-degree Taylor polynomial for f about $x = 0$.

$$\sum_{n=0}^{3} \frac{f^n(0)x^n}{n!}$$

$$3 - 2x + \frac{3x^2}{2!} - \frac{23x^3}{2 \cdot 3!}$$

(b) Write the first three nonzero terms of the Maclaurin series for e^x. Write the second-degree Taylor polynomial for $e^x f(x)$ about $x = 0$.

$$1 + x + \frac{x^2}{2}$$

$$\left(1+x+\frac{x^2}{2}\right)\left(3-2x+\frac{3x^2}{2}-\frac{23x^3}{12}\right)$$

$$3 - 2x + \frac{3x^2}{2} + 3x - 2x^2 + \frac{3x^2}{2} + \ldots$$

$$3 + x + x^2\left(\frac{3}{2} - 2 + \frac{3}{2}\right)$$

$$3 + x + x^2$$

-20-

Continue question 6 on page 2

6A
2 of 2

(c) Let h be the function defined by $h(x) = \int_0^x f(t)\,dt$. Use the Taylor polynomial found in part (a) to find an approximation for $h(1)$.

$h(x) = 3x - x^2 + \frac{x^3}{2} - \frac{23x^4}{8 \cdot 3!}$

$h(1) = 3 - 1 + \frac{1}{2} - \frac{23}{8 \cdot 3!}$

$= 2 + \frac{1}{2} - \frac{23}{48}$

$= \frac{96 + 24 - 23}{48}$

$= \frac{97}{48}$

(d) It is known that the Maclaurin series for h converges to $h(x)$ for all real numbers x. It is also known that the individual terms of the series for $h(1)$ alternate in sign and decrease in absolute value to 0. Use the alternating series error bound to show that the approximation found in part (c) differs from $h(1)$ by at most 0.45.

$T_4(x) = \int_0^x \frac{54t^4}{4!}\,dt = \frac{54x^5}{5!}$

$T_4(1) = \frac{54}{5!} = \frac{54}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{54}{20 \cdot 6} = \frac{54}{120} = \frac{27}{60} = \frac{9}{20}$

error bound = fourth term = $\frac{9}{20} = 0.45$

$\frac{9}{20} \leq 0.45$

error ≤ 0.45

NO CALCULATOR ALLOWED

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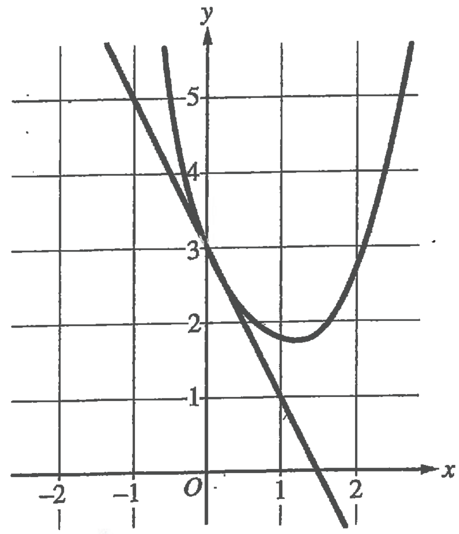


n	$f^{(n)}(0)$
2	3
3	$-\frac{23}{2}$
4	54

6. A function f has derivatives of all orders for all real numbers x. A portion of the graph of f is shown above, along with the line tangent to the graph of f at $x = 0$. Selected derivatives of f at $x = 0$ are given in the table above.

(a) Write the third-degree Taylor polynomial for f about $x = 0$.

$$P_3 = \frac{3x^0}{0!} + \frac{-2x^1}{1!} + \frac{3x^2}{2!} - \frac{23x^3}{2 \cdot 3!}$$

$$= 3 - 2x + \frac{3}{2}x^2 - \frac{23}{12}x^3$$

(b) Write the first three nonzero terms of the Maclaurin series for e^x. Write the second-degree Taylor polynomial for $e^x f(x)$ about $x = 0$.

$$1 + x + \frac{x^2}{2}$$

$$3 - 2x + \frac{3}{2}x^2$$
$$\times \; 1 + x + \frac{x^2}{2}$$
$$3 - 2x + \frac{3}{2}x^2$$
$$3x - 2x^2$$
$$\frac{3}{2}x^2$$

$$\boxed{P_2 = 3 + x + x^2}$$

Continue question 6 on page 21.

NO CALCULATOR ALLOWED

6B 2 of 2

(c) Let h be the function defined by $h(x) = \int_0^x f(t)\,dt$. Use the Taylor polynomial found in part (a) to find an approximation for $h(1)$.

$h(x) = \int_0^x 3 - 2t + \frac{3}{2}t^2 - \frac{23}{36}t^3\,dt$

$= 3x - x^2 + \frac{1}{2}x^3 - \frac{23}{36}x^4$

$h(1) = 3 - 1 + \frac{1}{2} - \frac{23}{36}$

$= \frac{5}{2} - \frac{23}{36}$

$= \frac{67}{36}$

$\begin{array}{r} {}^{4}18 \\ \times\ 5 \\ \hline 90 \\ -23 \\ \hline 67 \end{array}$

(d) It is known that the Maclaurin series for h converges to $h(x)$ for all real numbers x. It is also known that the individual terms of the series for $h(1)$ alternate in sign and decrease in absolute value to 0. Use the alternating series error bound to show that the approximation found in part (c) differs from $h(1)$ by at most 0.45.

$4! = 24 = 4 \cdot 3 \cdot 2$

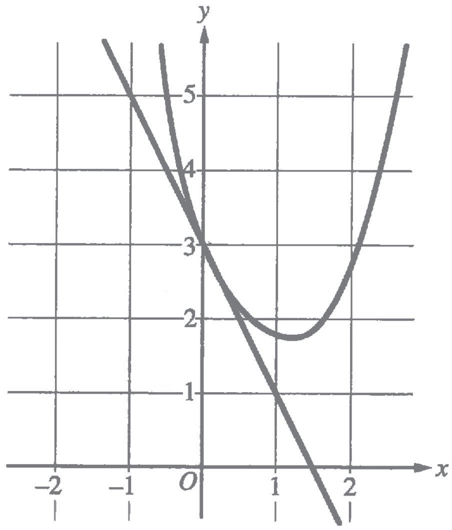
$\frac{f^{(4)}(0)\,x^4}{4!} = \frac{54\,x^4}{4!} = \frac{27x^4}{12}$

$\frac{\frac{27}{5}x^5}{12} = \frac{27x^5}{60}$

$\frac{27}{60} \le 0.45$

GO ON TO THE NEXT PAGE.

NO CALCULATOR ALLOWED



n	$f^{(n)}(0)$
2	3
3	$-\frac{23}{2}$
4	54

6. A function f has derivatives of all orders for all real numbers x. A portion of the graph of f is shown above, along with the line tangent to the graph of f at $x = 0$. Selected derivatives of f at $x = 0$ are given in the table above.

(a) Write the third-degree Taylor polynomial for f about $x = 0$.

$$f(0) + f'(0)(x) + \frac{f''(0)(x)^2}{2!} + \frac{f'''(0)(x)^3}{3!} = T_3(x)$$

$$3 + \left(\frac{5-3}{-1-0}\right)x + \frac{3x^2}{2} + -\frac{23}{2}\cdot\frac{x^3}{6} = T_3(x)$$

$$\boxed{3 - 2x + \frac{3}{2}x^2 - \frac{23}{12}x^3 = T_3(x)}$$

(b) Write the first three nonzero terms of the Maclaurin series for e^x. Write the second-degree Taylor polynomial for $e^x f(x)$ about $x = 0$.

$$M(x) = 1 + x + \frac{x^2}{2}$$

$$T_2(x) = f(0)\left(1 + x + \frac{x^2}{2}\right) = 3\left(1 + x + \frac{x^2}{2}\right)$$

$$\boxed{T_2(x) = 3 + 3x + \frac{3}{2}x^2}$$

Continue question 6 on page 21.

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(c) Let h be the function defined by $h(x) = \int_0^x f(t)\,dt$. Use the Taylor polynomial found in part (a) to find an approximation for $h(1)$.

$h(1) = \int_0^1 f(t)\,dt$

$h(1) \approx 3 - 2(1) + \frac{3}{2}(1)^2 - \frac{23}{12}(1)^3$

$h(1) \approx 1 + \frac{3}{2} - \frac{23}{12}$

$h(1) \approx \frac{30}{12} - \frac{23}{12}$

$\boxed{h(1) \approx \frac{7}{12}}$

(d) It is known that the Maclaurin series for h converges to $h(x)$ for all real numbers x. It is also known that the individual terms of the series for $h(1)$ alternate in sign and decrease in absolute value to 0. Use the alternating series error bound to show that the approximation found in part (c) differs from $h(1)$ by at most 0.45.

$\frac{54x^4}{24} \leq 0.45$

Question 6

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

In this problem a function f is presented that has derivatives of all orders for all real numbers x. A figure showing a portion of the graph of f and a line tangent to the graph of f at $x = 0$ is given, as is a table showing values for $f^{(2)}(0)$, $f^{(3)}(0)$, and $f^{(4)}(0)$.

In part (a) students were asked to write the third-degree Taylor polynomial for f about $x = 0$. A response should demonstrate that terms of the Taylor polynomial have the form $\frac{f^{(n)}(0)}{n!}x^n$, determine $f(0)$ and $f'(0)$ from the given graph, and find values for $f''(0) = f^{(2)}(0)$ and $f^{(3)}(0)$ in the table to construct the requested Taylor polynomial.

In part (b) students were asked to write the first three nonzero terms of the Maclaurin series for e^x and to provide the second-degree Taylor polynomial for $e^x f(x)$ about $x = 0$. A response should state that the Maclaurin series for e^x starts with the terms $1 + x + \frac{1}{2!}x^2 + \cdots$ and then form the second-degree Taylor polynomial for $e^x f(x)$ about $x = 0$ using the terms of degree at most 2 in the product $\left(1 + x + \frac{1}{2!}x^2\right) \cdot T_3(x)$, where $T_3(x)$ is the Taylor polynomial found in part (a).

In part (c) students were asked to use the Taylor polynomial found in part (a) to approximate $h(1)$, where $h(x) = \int_0^x f(t)\,dt$. A response should demonstrate that $h(1) = \int_0^1 f(t)\,dt \approx \int_0^1 T_3(t)\,dt$ where $T_3(x)$ is the Taylor polynomial found in part (a). $\int_0^1 T_3(t)\,dt$ should be evaluated using antidifferentiation and the Fundamental Theorem of Calculus.

In part (d) it is given that the Maclaurin series for h converges to $h(x)$ everywhere and that the individual terms of the series for $h(1)$ alternate in sign and decrease in absolute value to 0. Students were asked to use the alternating series error bound to show that the approximation found in part (c) differs from $h(1)$ by at most 0.45. A response should demonstrate that the error in the approximation is bounded by the magnitude of the first omitted term of the series for $h(1)$. This term is found by integrating the fourth-degree term of the Taylor series for f about $x = 0$ across the interval $[0, 1]$. Computing this term demonstrates the desired error bound.

For part (a) see LO LIM-8.A/EK LIM-8.A.1. For part (b) see LO LIM-8.F/EK LIM-8.F.2, LO LIM-8.G/EK LIM.8.G.1. For part (c) see LO LIM-8.G/EK LIM-8.G.1. For part (d) see LO LIM-8.C/EK LIM-8.C.2. This problem incorporates the following Mathematical Practices: Practice 1: Implementing Mathematical Processes, Practice 2: Connecting Representations, and Practice 4: Communication and Notation.

Question 6 (continued)

Sample: 6A
Score: 9

The response earned 9 points: 2 points in part (a), 2 points in part (b), 2 points in part (c), and 3 points in part (d). In part (a) the response earned the first point in line 2 for the first two terms, $3 - 2x$, of the third-degree Taylor polynomial for f. The response earned the second point in line 2 for the remaining terms, $+\frac{3x^2}{2!} - \frac{23x^3}{2 \cdot 3!}$, of the third-degree Taylor polynomial for f. In part (b) the response earned the first point in line 1 for the first three nonzero terms of the Maclaurin series for e^x, $1 + x + \frac{x^2}{2}$. The response earned the second point in line 5 with the second-degree Taylor polynomial for $e^x f(x)$ about $x = 0$, $3 + x + x^2$, with supporting work. In part (c) the response earned the first point in line 1 with the correct antiderivative $3x - x^2 + \frac{x^3}{2} - \frac{23x^4}{8 \cdot 3!}$. Numerical simplification is not required. The response would have earned the second point in line 2 with the evaluation $h(1) = 3 - 1 + \frac{1}{2} - \frac{23}{8 \cdot 3!}$ and no numerical simplification. The response simplifies to $\frac{97}{48}$ in line 5 and earned the second point. In part (d) the response earned the first point in line 1 with $\int_0^x \frac{54t^4}{4!}\, dt$. The response would have earned the second point in line 2 with $\frac{54}{5!}$ without simplification. The second point was earned with a correct simplification of $\frac{9}{20}$ at the end of line 2. The response earned the third point in lines 3 and 5 with "$\frac{9}{20} = 0.45$" and "error ≤ 0.45."

Sample: 6B
Score: 6

The response earned 6 points: 2 points in part (a), 2 points in part (b), no points in part (c), and 2 points in part (d). In part (a) the response earned the first point in line 2 for the first two terms, $3 - 2x$, of the third-degree Taylor polynomial for f. The response earned the second point in line 2 for the remaining terms, $+\frac{3}{2}x^2 - \frac{23}{12}x^3$, of the third-degree Taylor polynomial for f. In part (b) the response earned the first point in line 1 on the left for the first three nonzero terms of the Maclaurin series for e^x, $1 + x + \frac{x^2}{2}$. The response earned the second point in the last line with the boxed second-degree Taylor polynomial for $e^x f(x)$ about $x = 0$, $P_2 = 3 + x + x^2$, with supporting work. In part (c) the response did not earn the first point because the response has a copy error in the expression for $f(x)$ in the fourth term of the integrand, $-\frac{23}{36}t^3$, in line 1. The missing parentheses in the integrand do not impact earning the point. The response also has an antidifferentiation error in line 2 in the fourth-degree term (a missing factor of 4 in the denominator). Because the response has two errors, the response is not eligible for the second point. In part (d) the response would have earned the first point in line 2 with $\frac{\frac{27}{5}x^5}{12}$. The

Question 6 (continued)

first point was earned with the correct simplification, $\frac{27x^5}{60}$. The response earned the second point in line 3 with $\frac{27}{60}$. The response did not earn the third point because there is no reference to the error being bounded.

Sample: 6C
Score: 3

The response earned 3 points: 2 points in part (a), 1 point in part (b), no points in part (c), and no points in part (d). In part (a) the response earned the first point in line 3 for the first two terms, $3 - 2x$, of the third-degree Taylor polynomial for f. The response earned the second point in line 3 for the remaining terms, $+\frac{3}{2}x^2 - \frac{23}{12}x^3$, of the third-degree Taylor polynomial for f. In part (b) the response earned the first point in line 1 for the first three nonzero terms of the Maclaurin series for e^x, $M(x) = 1 + x + \frac{x^2}{2}$. The response did not earn the second point because the Taylor polynomial for $e^x f(x)$, $3\left(1 + x + \frac{x^2}{2}\right)$, in line 2 is incorrect. In part (c) the response did not earn the first point because the third-degree Taylor polynomial found in part (a) is not antidifferentiated. Because the response does not antidifferentiate the third-degree Taylor polynomial from part (a), the response is not eligible for the second point. In part (d) the response did not earn the first point because there is no attempt to antidifferentiate the fourth-degree term of f. Without an attempt to antidifferentiate the fourth-degree term of f, the response is not eligible to earn either the second or the third point.